Algorithms: DFS, STRONGLY CONNECTED COMPONENTS, FLOWS

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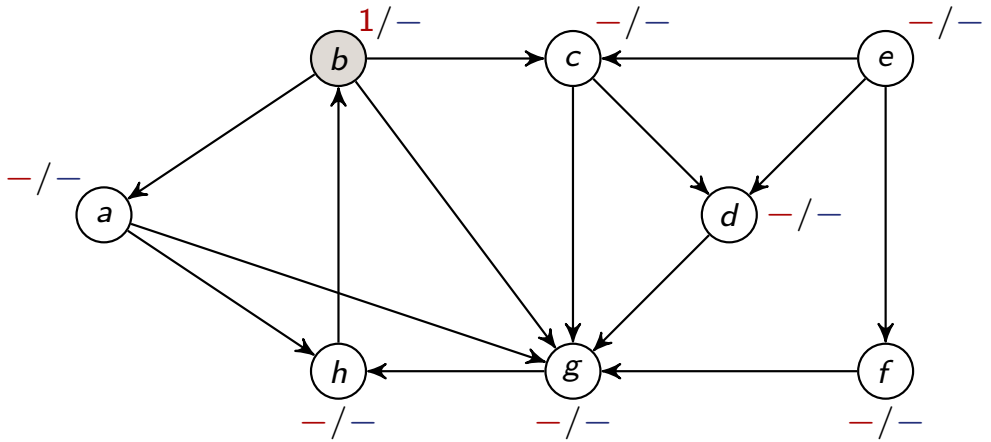
Lecture 15, 09.04.2024

Pseudocode of DFS

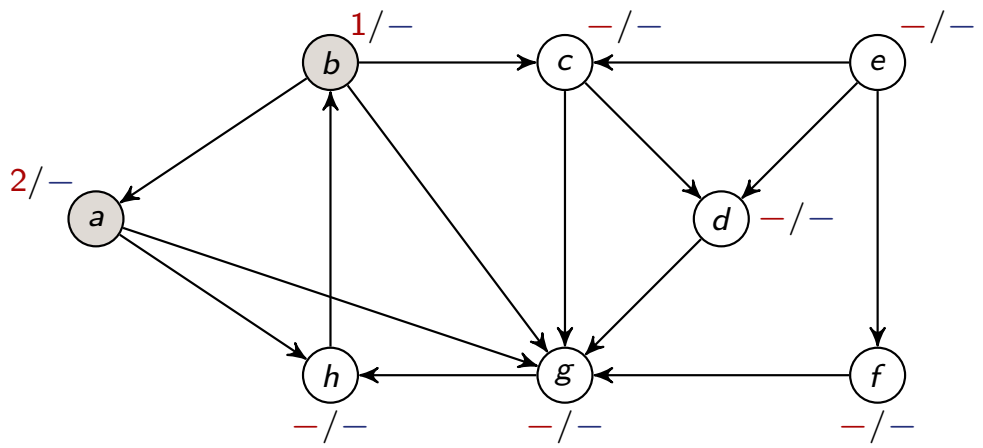
```
DFS-VISIT(G, u)
  time = time + 1
  u.d = time
  u.color = GRAY                    // discover u
  for each v ∈ G.Adj[u]             // explore (u, v)
      if v.color == WHITE
          DFS-VISIT(v)
  u.color = BLACK
  time = time + 1
  u.f = time                        // finish u
```



time = 1

Pseudocode of DFS

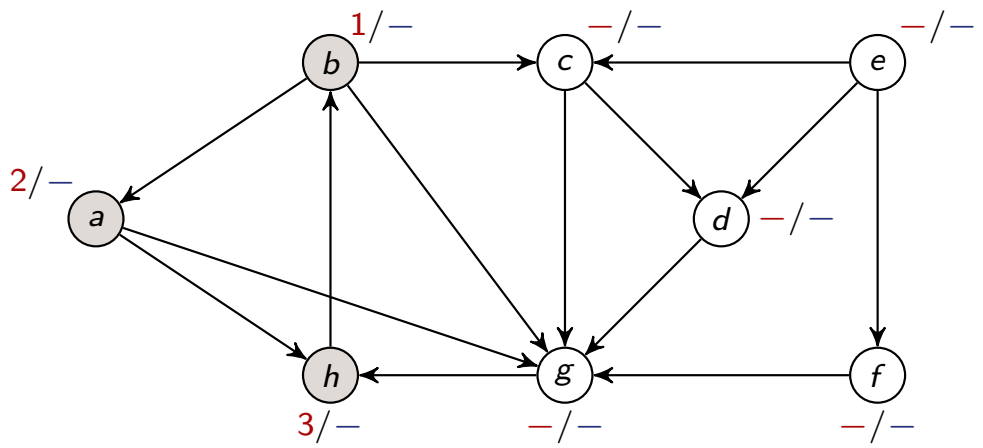
```
DFS-VISIT(G, u)
  time = time + 1
  u.d = time
  u.color = GRAY                    // discover u
  for each v ∈ G.Adj[u]             // explore (u, v)
      if v.color == WHITE
          DFS-VISIT(v)
  u.color = BLACK
  time = time + 1
  u.f = time                        // finish u
```



time = 2

Pseudocode of DFS

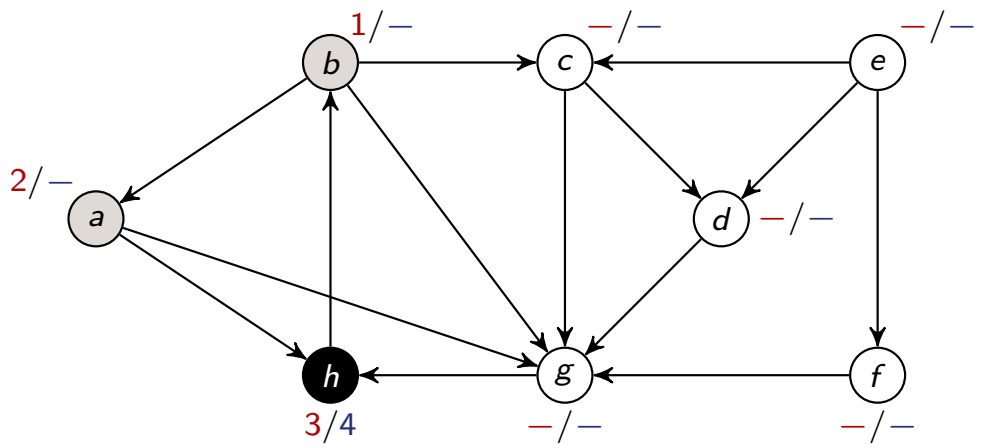
```
DFS-VISIT(G, u)
  time = time + 1
  u.d = time
  u.color = GRAY                    // discover u
  for each v ∈ G.Adj[u]             // explore (u, v)
      if v.color == WHITE
          DFS-VISIT(v)
  u.color = BLACK
  time = time + 1
  u.f = time                        // finish u
```



time = 3

Pseudocode of DFS

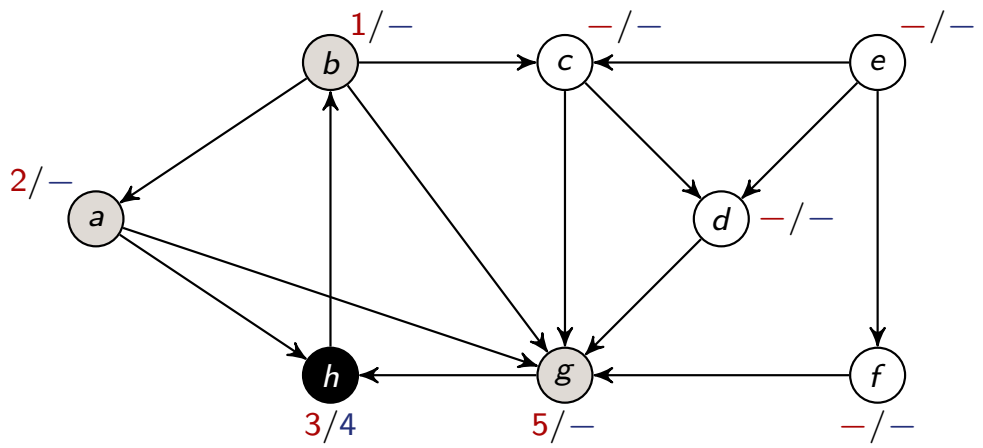
```
DFS-VISIT(G, u)
  time = time + 1
  u.d = time
  u.color = GRAY                    // discover u
  for each v ∈ G.Adj[u]             // explore (u, v)
      if v.color == WHITE
          DFS-VISIT(v)
  u.color = BLACK
  time = time + 1
  u.f = time                        // finish u
```



time = 4

Pseudocode of DFS

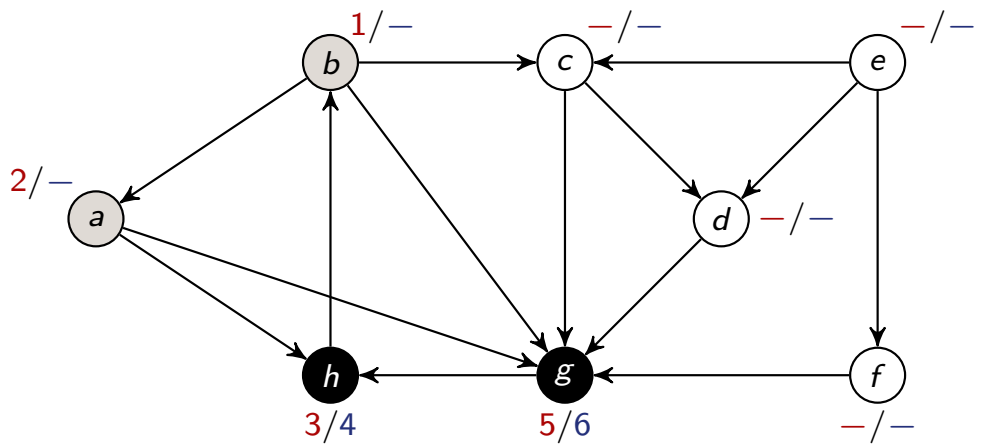
```
DFS-VISIT(G, u)
  time = time + 1
  u.d = time
  u.color = GRAY                    // discover u
  for each v ∈ G.Adj[u]             // explore (u, v)
      if v.color == WHITE
          DFS-VISIT(v)
  u.color = BLACK
  time = time + 1
  u.f = time                        // finish u
```



time = 5

Pseudocode of DFS

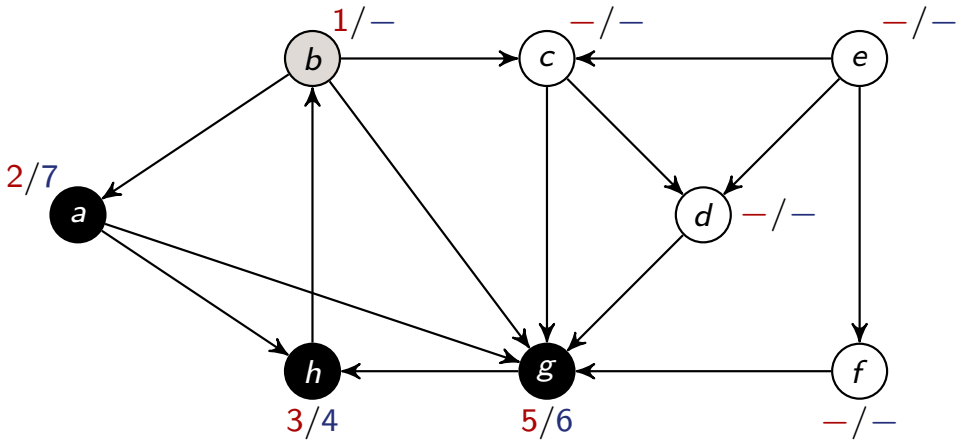
```
DFS-VISIT(G, u)
  time = time + 1
  u.d = time
  u.color = GRAY                    // discover u
  for each v ∈ G.Adj[u]             // explore (u, v)
      if v.color == WHITE
          DFS-VISIT(v)
  u.color = BLACK
  time = time + 1
  u.f = time                        // finish u
```



time = 6

Pseudocode of DFS

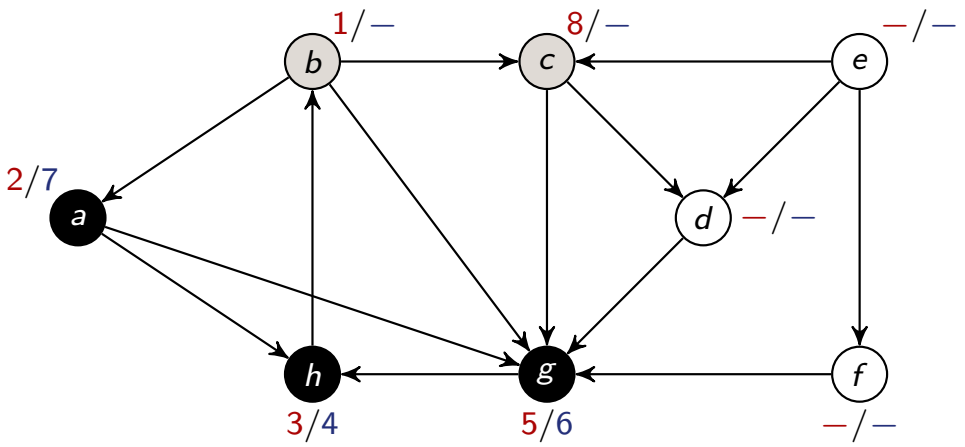
```
DFS-VISIT(G, u)
  time = time + 1
  u.d = time
  u.color = GRAY                    // discover u
  for each v ∈ G.Adj[u]             // explore (u, v)
      if v.color == WHITE
          DFS-VISIT(v)
  u.color = BLACK
  time = time + 1
  u.f = time                        // finish u
```



time = 7

Pseudocode of DFS

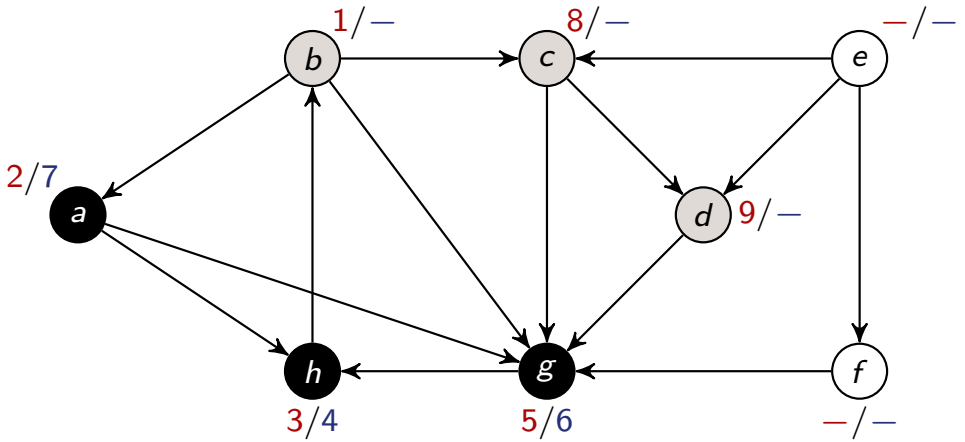
```
DFS-VISIT(G, u)
  time = time + 1
  u.d = time
  u.color = GRAY                    // discover u
  for each v ∈ G.Adj[u]             // explore (u, v)
      if v.color == WHITE
          DFS-VISIT(v)
  u.color = BLACK
  time = time + 1
  u.f = time                        // finish u
```



time = 8

Pseudocode of DFS

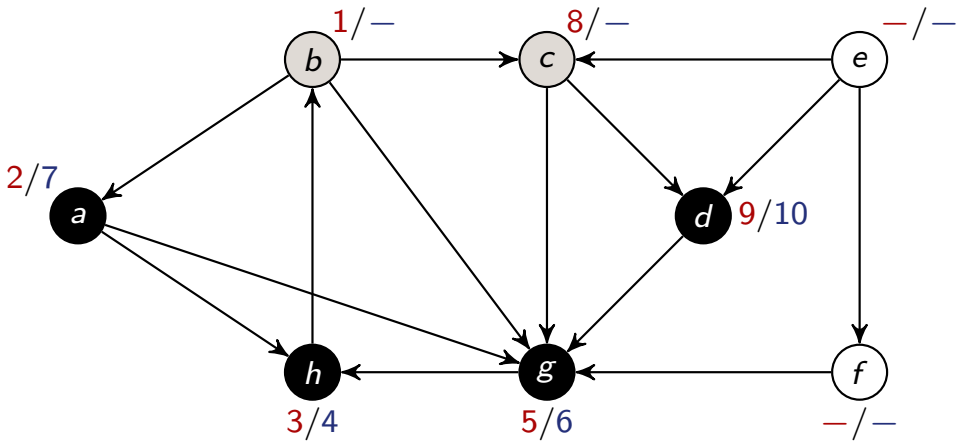
```
DFS-VISIT(G, u)
  time = time + 1
  u.d = time
  u.color = GRAY                    // discover u
  for each v ∈ G.Adj[u]             // explore (u, v)
      if v.color == WHITE
          DFS-VISIT(v)
  u.color = BLACK
  time = time + 1
  u.f = time                        // finish u
```



time = 9

Pseudocode of DFS

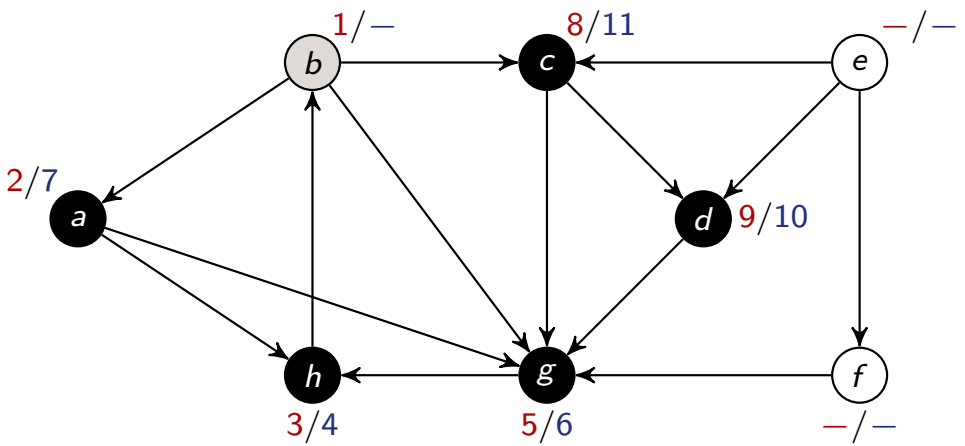
```
DFS-VISIT(G, u)
  time = time + 1
  u.d = time
  u.color = GRAY                    // discover u
  for each v ∈ G.Adj[u]             // explore (u, v)
      if v.color == WHITE
          DFS-VISIT(v)
  u.color = BLACK
  time = time + 1
  u.f = time                        // finish u
```



time = 10

Pseudocode of DFS

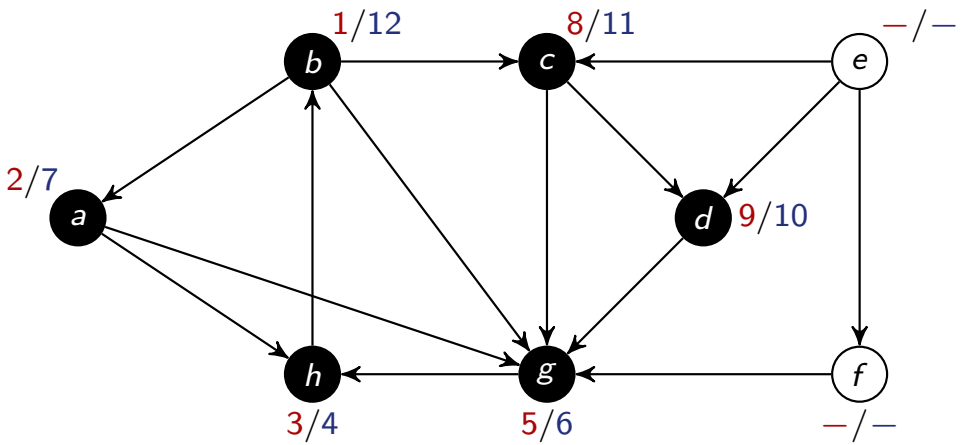
```
DFS-VISIT(G, u)
  time = time + 1
  u.d = time
  u.color = GRAY                    // discover u
  for each v ∈ G.Adj[u]             // explore (u, v)
      if v.color == WHITE
          DFS-VISIT(v)
  u.color = BLACK
  time = time + 1
  u.f = time                        // finish u
```



time = 11

Pseudocode of DFS

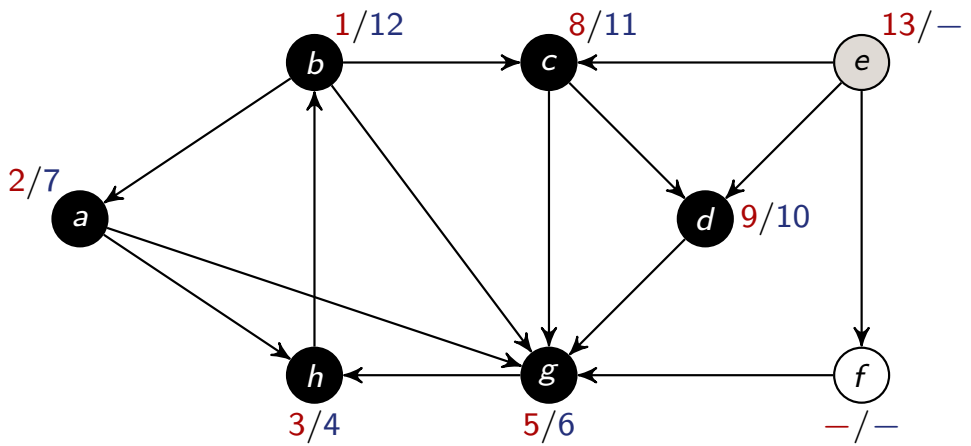
```
DFS-VISIT(G, u)
  time = time + 1
  u.d = time
  u.color = GRAY                    // discover u
  for each v ∈ G.Adj[u]             // explore (u, v)
      if v.color == WHITE
          DFS-VISIT(v)
  u.color = BLACK
  time = time + 1
  u.f = time                        // finish u
```



time = 12

Pseudocode of DFS

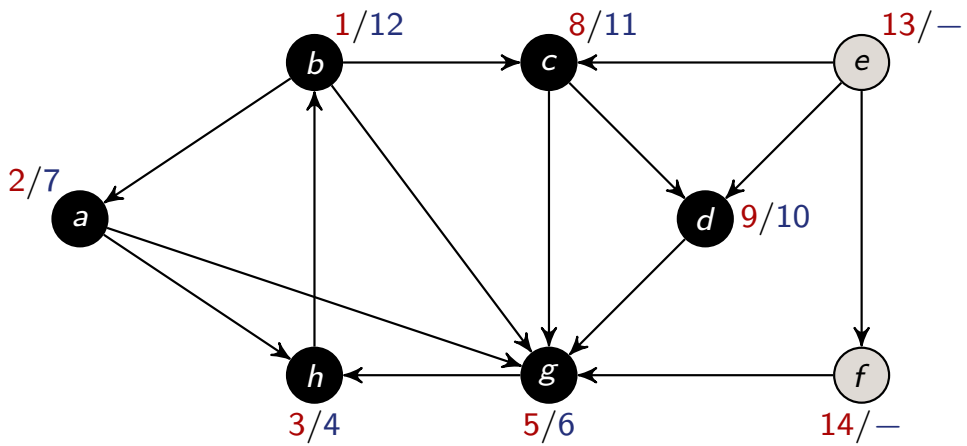
```
DFS-VISIT(G, u)
  time = time + 1
  u.d = time
  u.color = GRAY                    // discover u
  for each v ∈ G.Adj[u]             // explore (u, v)
      if v.color == WHITE
          DFS-VISIT(v)
  u.color = BLACK
  time = time + 1
  u.f = time                        // finish u
```



time = 13

Pseudocode of DFS

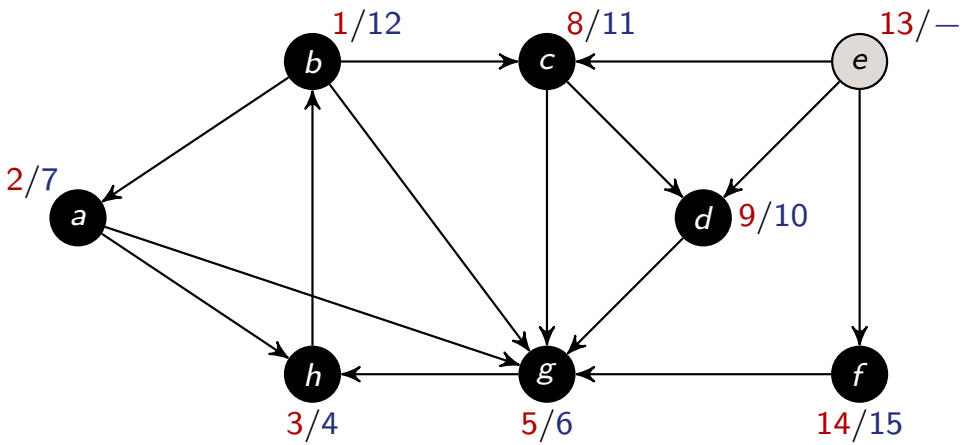
```
DFS-VISIT(G, u)
  time = time + 1
  u.d = time
  u.color = GRAY                    // discover u
  for each v ∈ G.Adj[u]             // explore (u, v)
      if v.color == WHITE
          DFS-VISIT(v)
  u.color = BLACK
  time = time + 1
  u.f = time                        // finish u
```



time = 14

Pseudocode of DFS

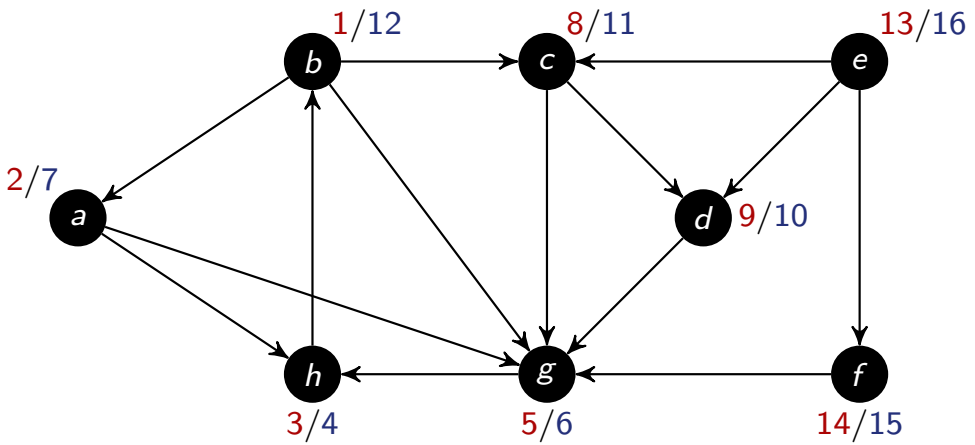
```
DFS-VISIT(G, u)
  time = time + 1
  u.d = time
  u.color = GRAY                    // discover u
  for each v ∈ G.Adj[u]             // explore (u, v)
      if v.color == WHITE
          DFS-VISIT(v)
  u.color = BLACK
  time = time + 1
  u.f = time                        // finish u
```



time = 15

Pseudocode of DFS

```
DFS-VISIT(G, u)
  time = time + 1
  u.d = time
  u.color = GRAY                    // discover u
  for each v ∈ G.Adj[u]             // explore (u, v)
      if v.color == WHITE
          DFS-VISIT(v)
  u.color = BLACK
  time = time + 1
  u.f = time                        // finish u
```



time = 16

Runtime Analysis

```
DFS(G)
  for each u ∈ G.V
      u.color = WHITE
  time = 0
  for each u ∈ G.V
      if u.color == WHITE
          DFS-VISIT(G, u)
```

```
DFS-VISIT(G, u)
  time = time + 1
  u.d = time
  u.color = GRAY                  // discover u
  for each v ∈ G.Adj[u]           // explore (u, v)
      if v.color == WHITE
          DFS-VISIT(v)
  u.color = BLACK
  time = time + 1
  u.f = time                      // finish u
```

Runtime Analysis

```
DFS(G)
  for each u ∈ G.V
      u.color = WHITE
  time = 0
  for each u ∈ G.V
      if u.color == WHITE
          DFS-VISIT(G, u)
```

```
DFS-VISIT(G, u)
  time = time + 1
  u.d = time
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  for each v ∈ G.Adj[u]           // explore (u, v)
      if v.color == WHITE
          DFS-VISIT(v)
  u.color = BLACK
  time = time + 1
  u.f = time                      // finish u
```

- Color each vertex white takes time $\Theta(V)$

Runtime Analysis

```
DFS(G)
  for each u ∈ G.V
      u.color = WHITE
  time = 0
  for each u ∈ G.V
      if u.color == WHITE
          DFS-VISIT(G, u)
```

```
DFS-VISIT(G, u)
  time = time + 1
  u.d = time
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      if v.color == WHITE
          DFS-VISIT(v)
  u.color = BLACK
  time = time + 1
  u.f = time                      // finish u
```

- Color each vertex white takes time $\Theta(V)$
- Note that DFS-VISIT is called once for each vertex (when it is colored gray from white)

Runtime Analysis

```
DFS(G)
  for each u ∈ G.V
      u.color = WHITE
  time = 0
  for each u ∈ G.V
      if u.color == WHITE
          DFS-VISIT(G, u)
```

```
DFS-VISIT(G, u)
  time = time + 1
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      if v.color == WHITE
          DFS-VISIT(v)
  u.color = BLACK
  time = time + 1
  u.f = time                      // finish u
```

- Color each vertex white takes time $\Theta(V)$
- Note that DFS-VISIT is called once for each vertex (when it is colored gray from white)
- When DFS-VISIT(u) is called the **for** loop runs at most {#neighbors of u} times

Runtime Analysis

```
DFS(G)
  for each u ∈ G.V
      u.color = WHITE
  time = 0
  for each u ∈ G.V
      if u.color == WHITE
          DFS-VISIT(G, u)
```

```
DFS-VISIT(G, u)
  time = time + 1
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  for each v ∈ G.Adj[u]           // explore (u, v)
      if v.color == WHITE
          DFS-VISIT(v)
  u.color = BLACK
  time = time + 1
  u.f = time                      // finish u
```

- Color each vertex white takes time $\Theta(V)$
- Note that DFS-VISIT is called once for each vertex (when it is colored gray from white)
- When DFS-VISIT(u) is called the **for** loop runs at most {#neighbors of u} times
- Therefore the total time DFS-VISIT is run is

$$\sum_{u \in V} \{\#\text{neighbors of } u\} = \Theta(E)$$

Runtime Analysis

```
DFS(G)
  for each u ∈ G.V
      u.color = WHITE
  time = 0
  for each u ∈ G.V
      if u.color == WHITE
          DFS-VISIT(G, u)
```

```
DFS-VISIT(G, u)
  time = time + 1
  u.d = time
  u.color = GRAY                  // discover u
  for each v ∈ G.Adj[u]           // explore (u, v)
      if v.color == WHITE
          DFS-VISIT(v)
  u.color = BLACK
  time = time + 1
  u.f = time                      // finish u
```

- Color each vertex white takes time $\Theta(V)$
- Note that DFS-VISIT is called once for each vertex (when it is colored gray from white)
- When DFS-VISIT(u) is called the **for** loop runs at most {#neighbors of u} times
- Therefore the total time DFS-VISIT is run is

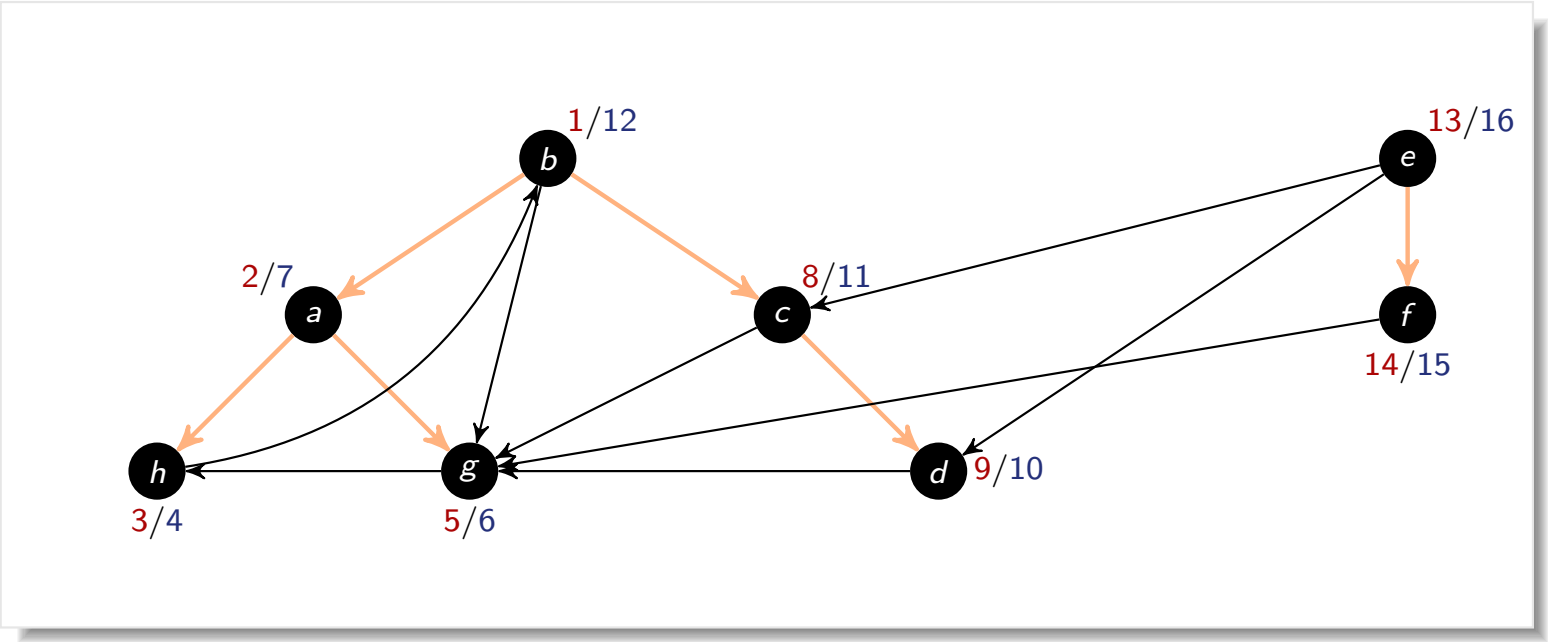
$$\sum_{u \in V} \{\#\text{neighbors of } u\} = \Theta(E)$$

Total Time: $\Theta(V + E)$

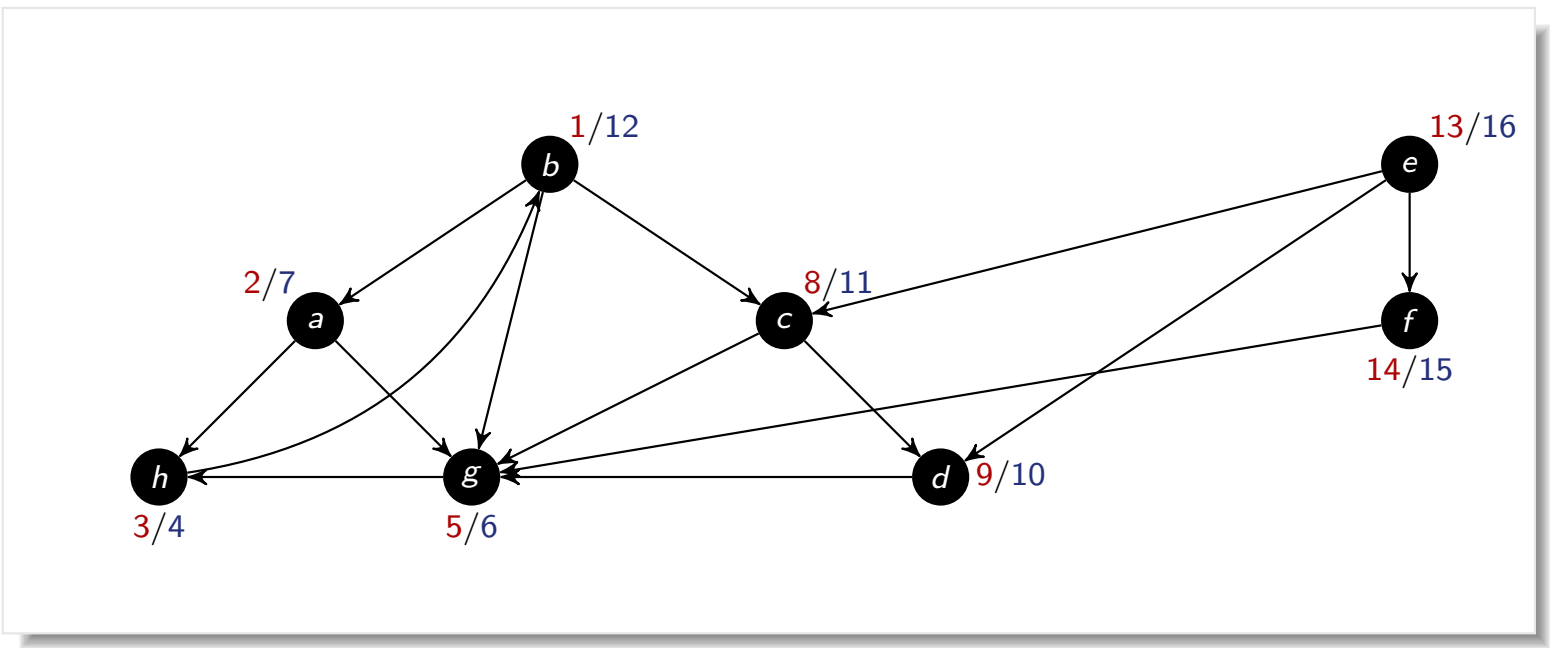
PROPERTIES OF DFS

Output of DFS

DFS forms a depth-first forest comprised of > 1 depth-first trees. Each tree is made of edges (u, v) such that u is gray and v is white when (u, v) is explored.

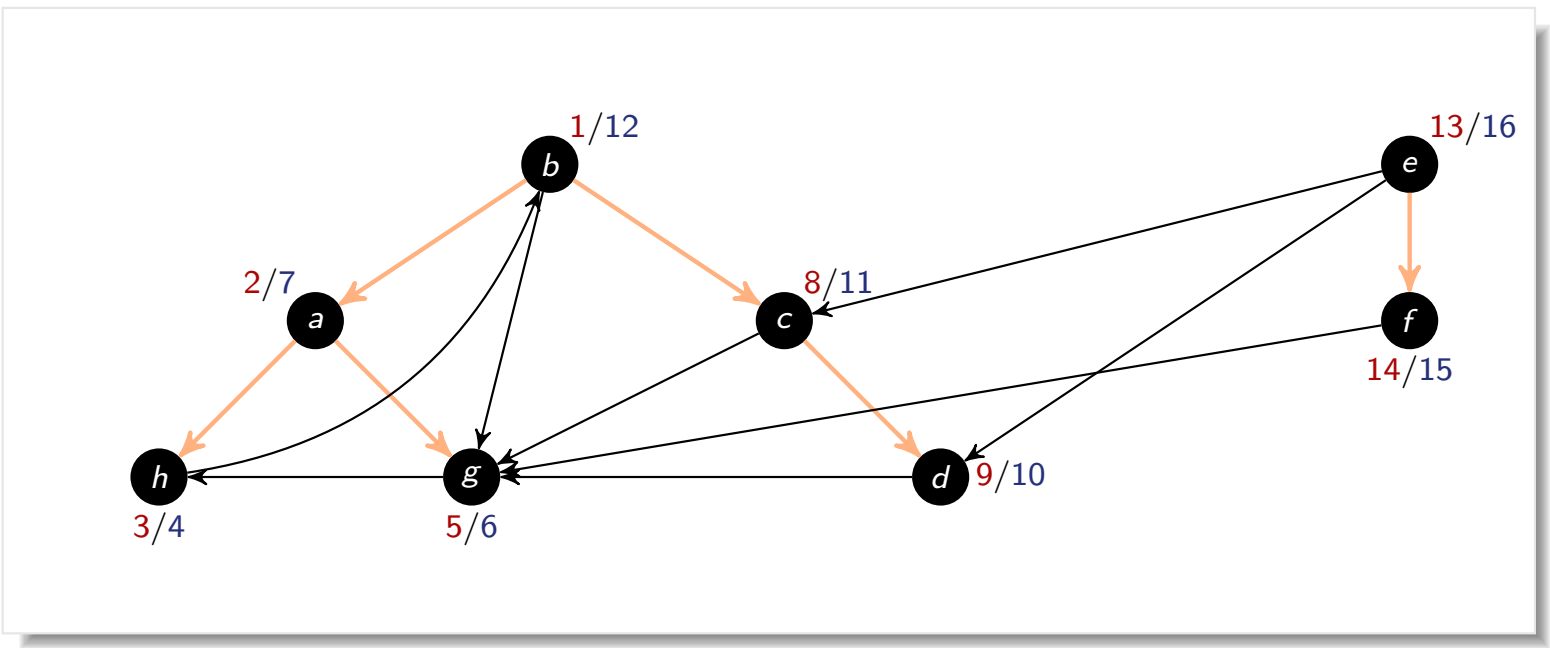


Classification of edges



Classification of edges

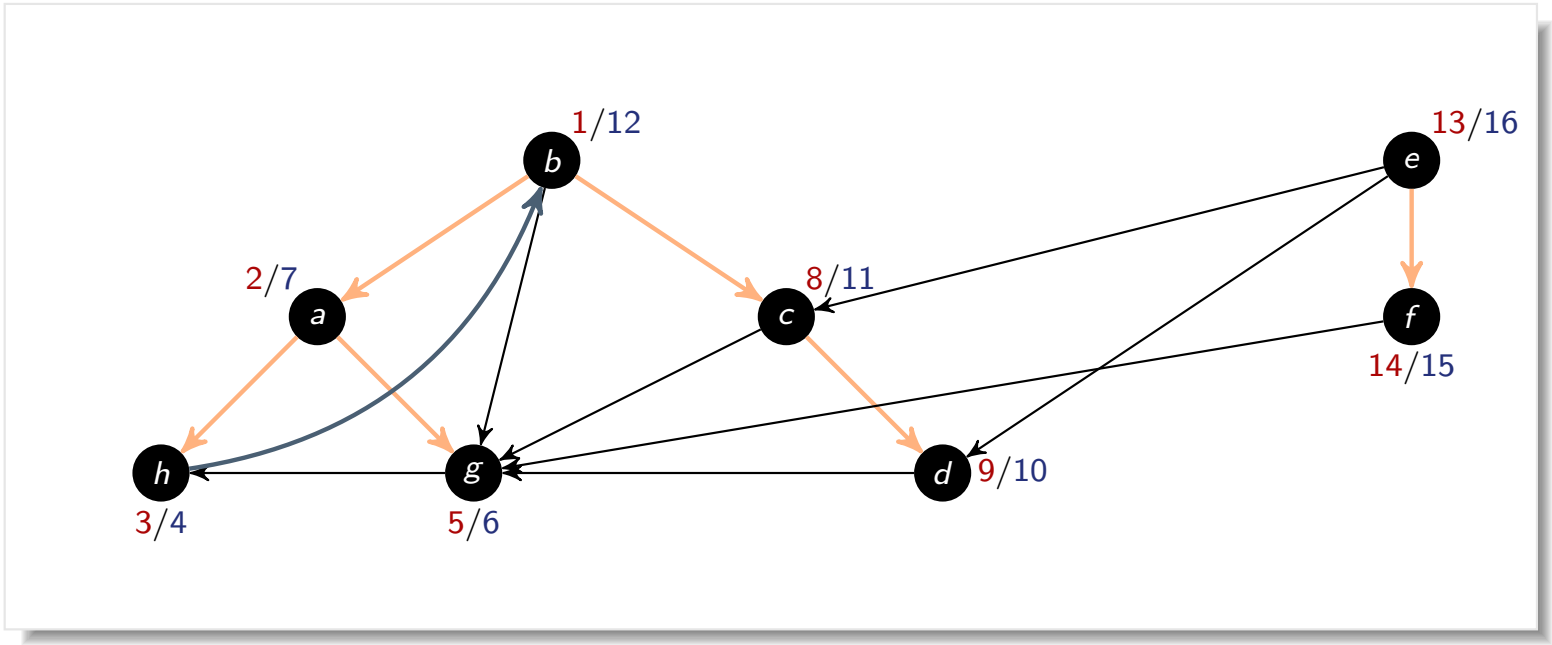
Tree edge: In the depth-first forest, found by exploring (u, v)



Classification of edges

Tree edge: In the depth-first forest, found by exploring (u, v)

Back edge: (u, v) where u is a descendant of v

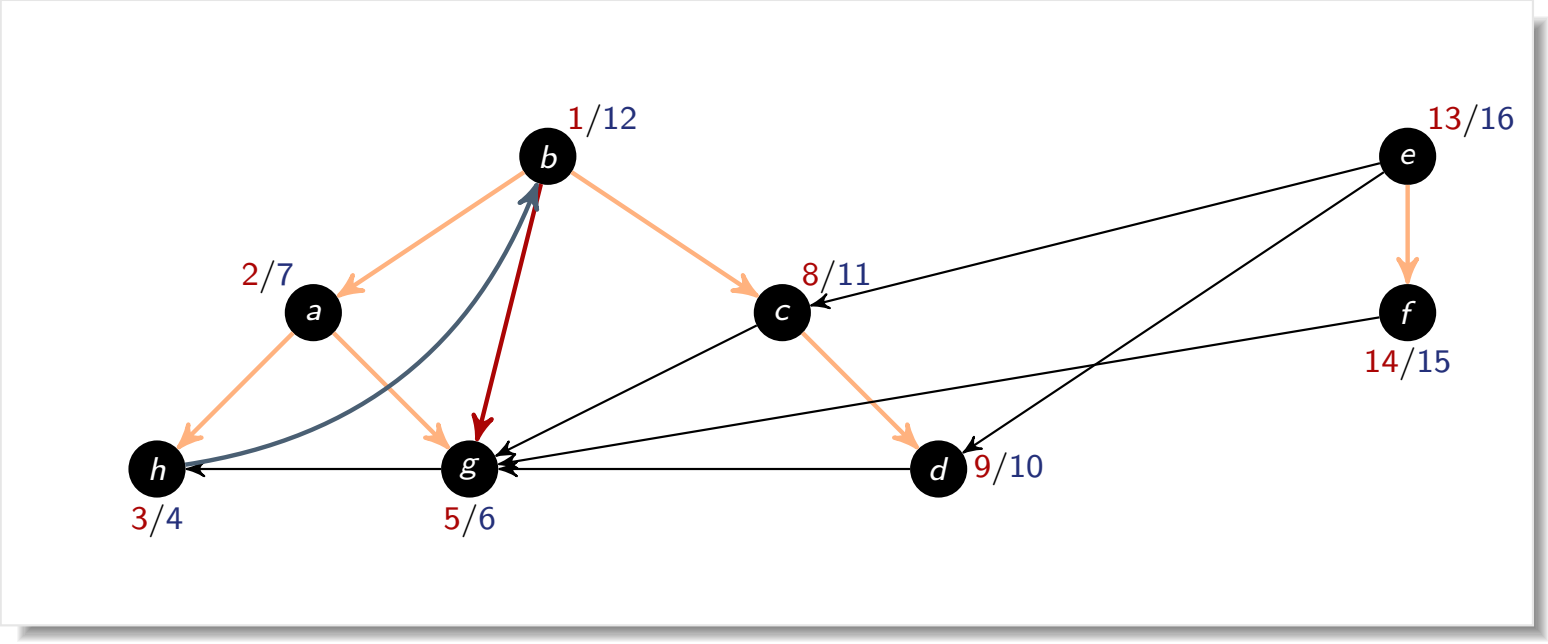


Classification of edges

Tree edge: In the depth-first forest, found by exploring (u, v)

Back edge: (u, v) where u is a descendant of v

Forward edge: (u, v) where v is a descendant of u, but not a tree edge



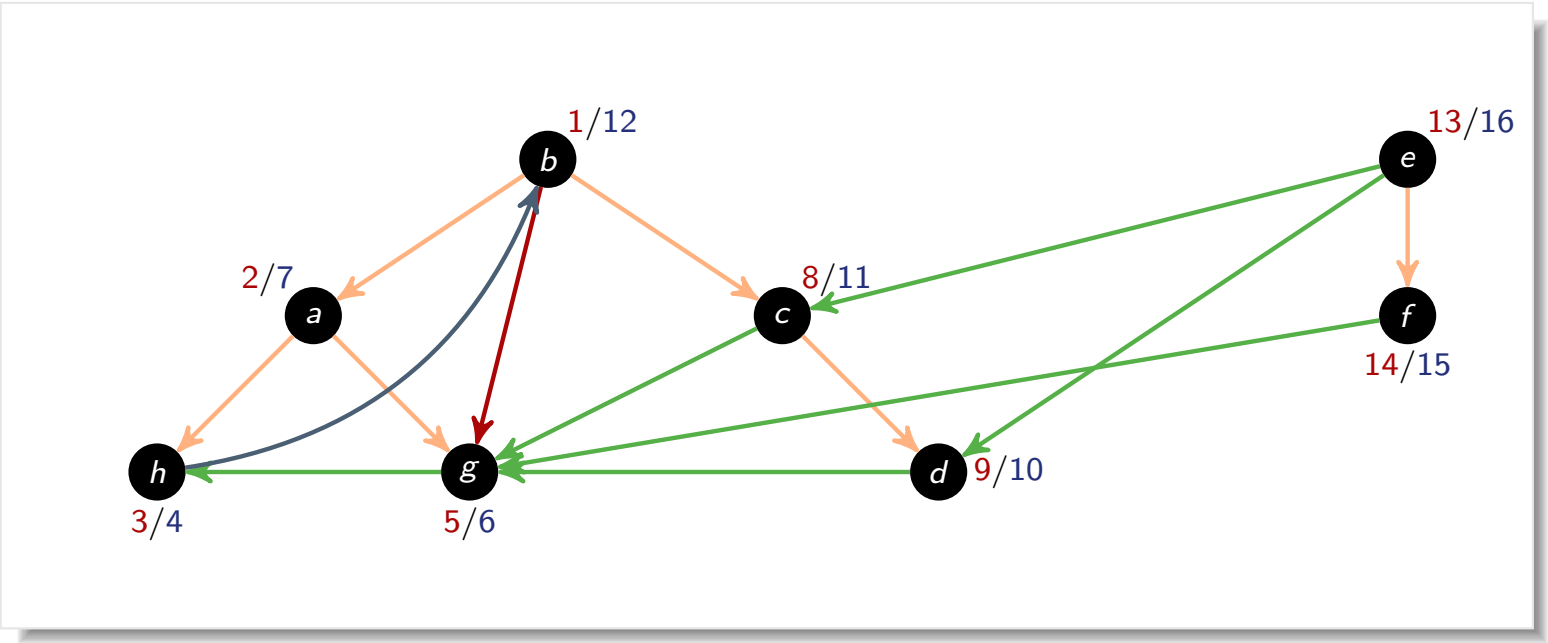
Classification of edges

Tree edge: In the depth-first forest, found by exploring (u, v)

Back edge: (u, v) where u is a descendant of v

Forward edge: (u, v) where v is a descendant of u, but not a tree edge

Cross edge: any other edge



Classification of edges

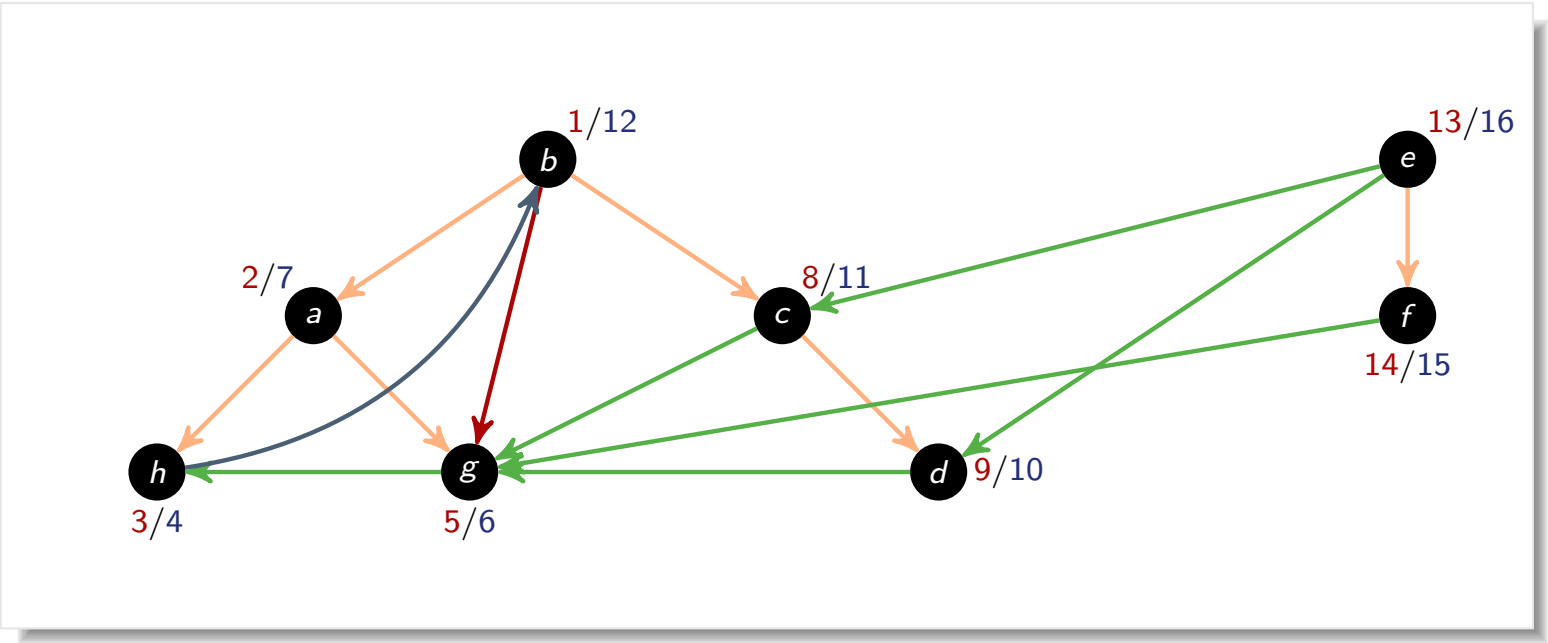
Tree edge: In the depth-first forest, found by exploring (u, v)

Back edge: (u, v) where u is a descendant of v

Forward edge: (u, v) where v is a descendant of u, but not a tree edge

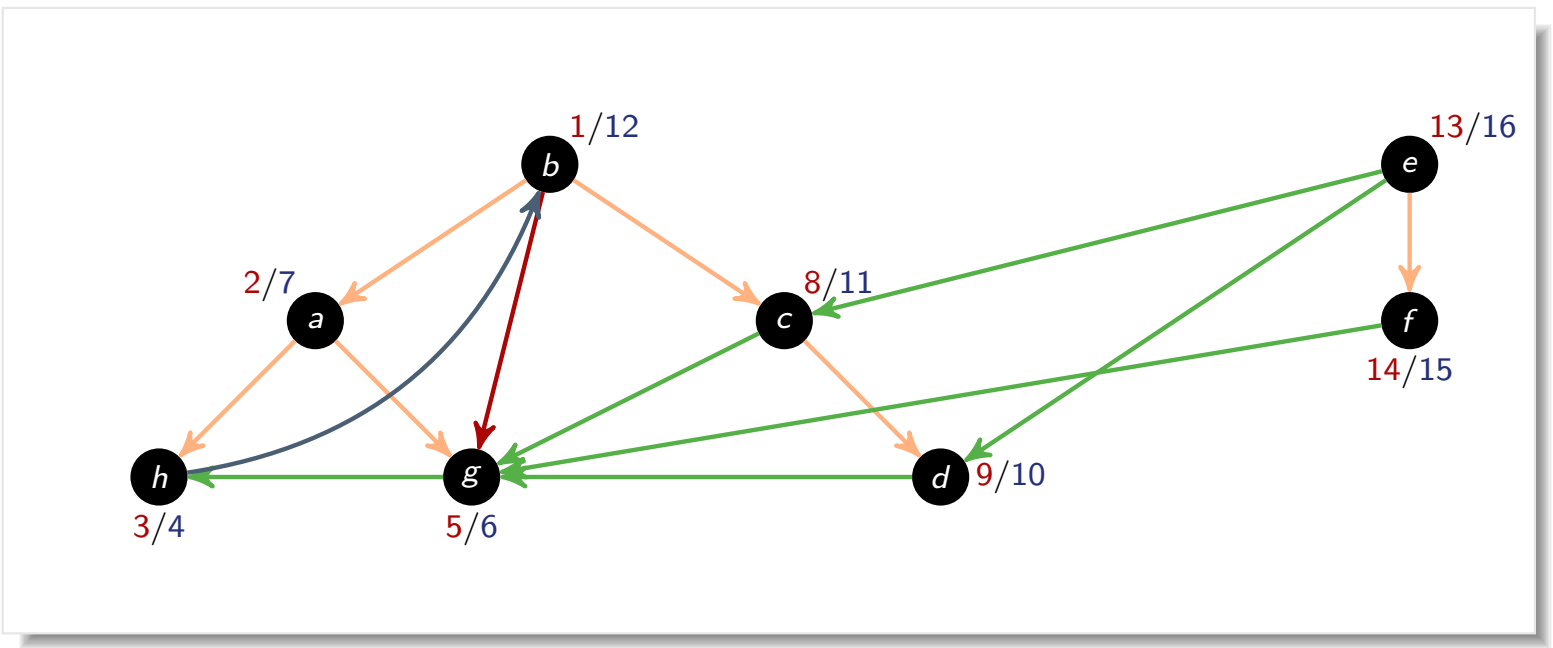
Cross edge: any other edge

In DFS of an undirected graph we get only tree and back/forward (we call them back) edges, no cross edges. Why?



Parenthesis theorem

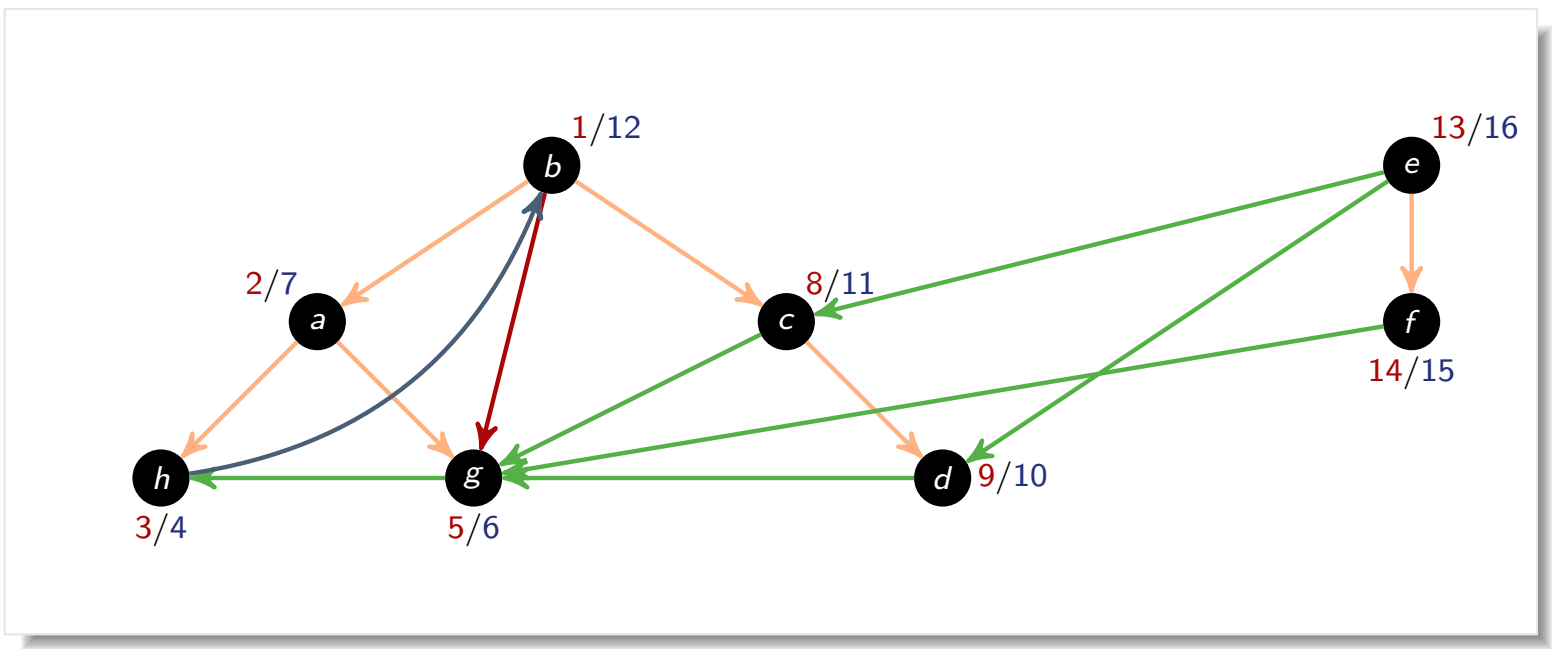
For all u, v exactly one of the following holds



Parenthesis theorem

For all u, v exactly one of the following holds

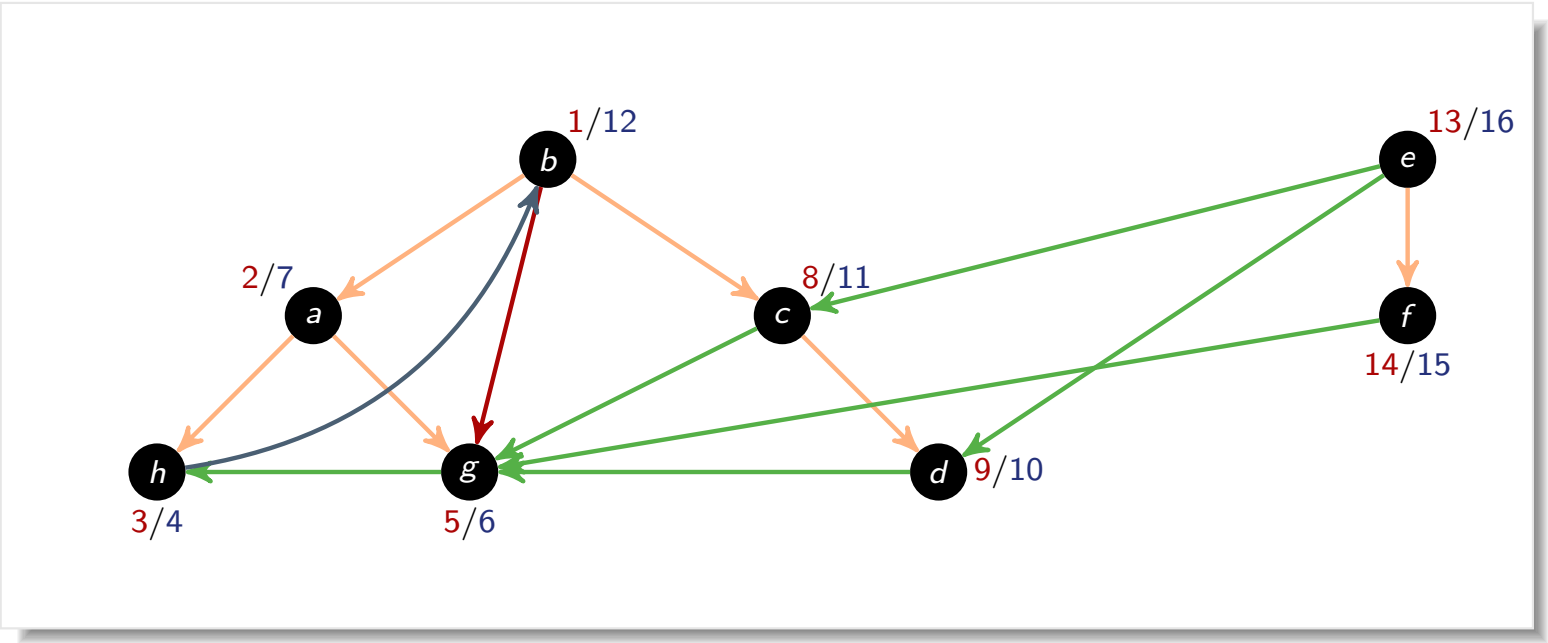
1. $[u.d, u.f]$ and $[v.d, v.f]$ are disjoint neither of u and v are descendant of each other



Parenthesis theorem

For all u, v exactly one of the following holds

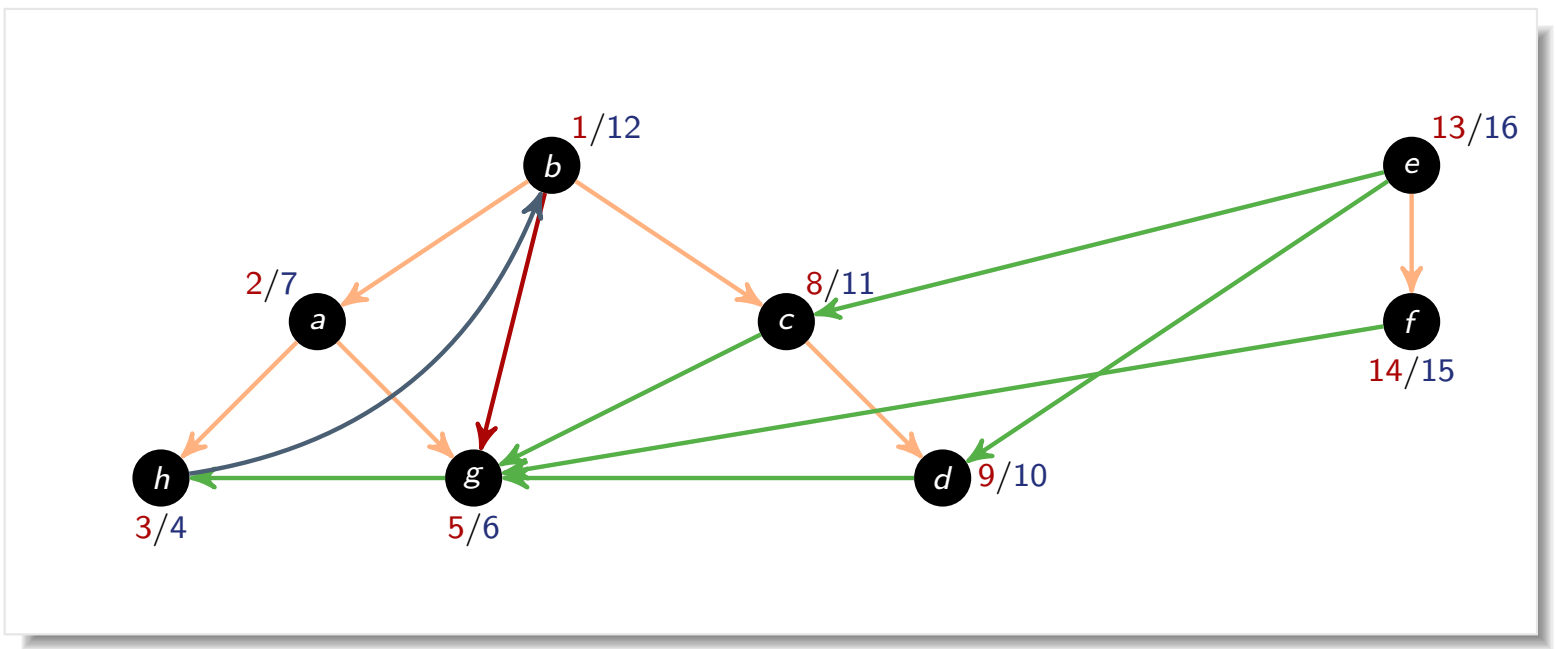
1. $[u.d, u.f]$ and $[v.d, v.f]$ are disjoint neither of u and v are descendant of each other
2. $u.d < v.d < v.f < u.f$ and v is a descendant of u



Parenthesis theorem

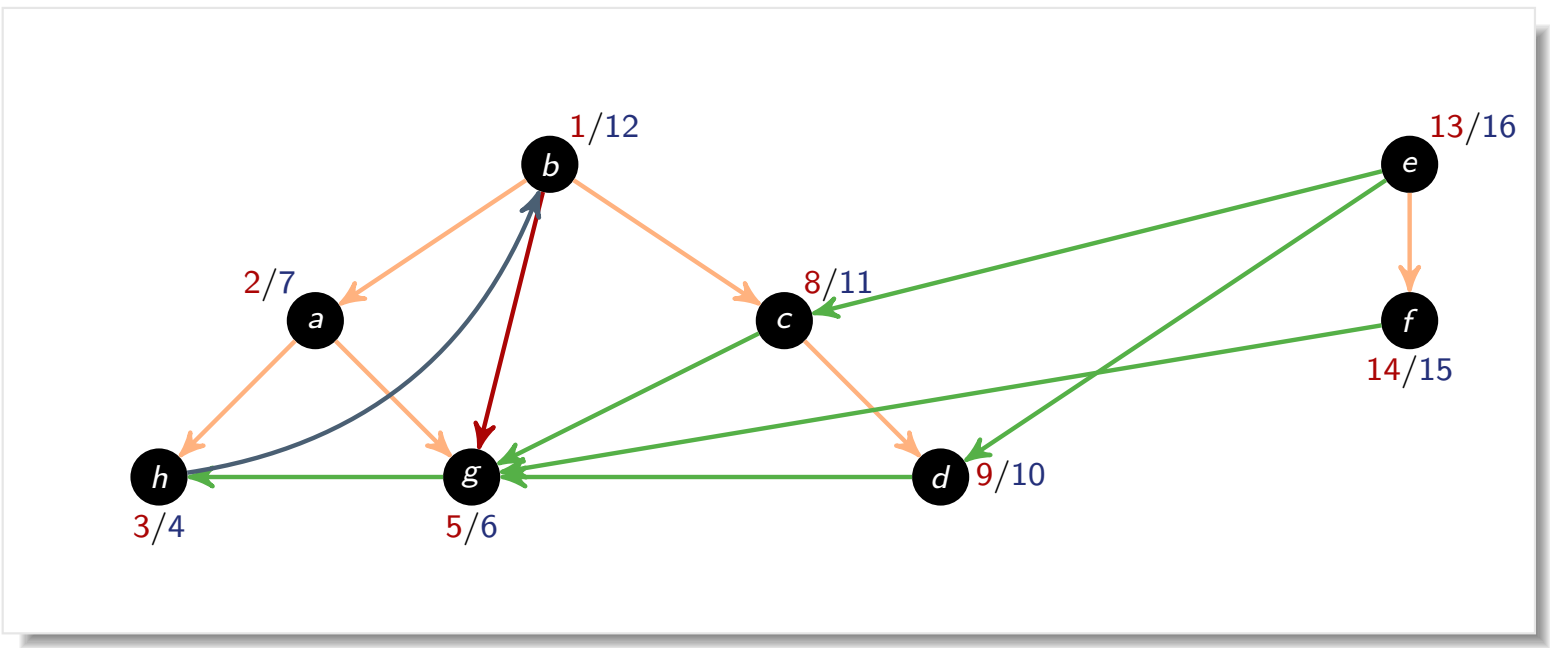
For all u, v exactly one of the following holds

1. $[u.d, u.f]$ and $[v.d, v.f]$ are disjoint neither of u and v are descendant of each other
2. $u.d < v.d < v.f < u.f$ and v is a descendant of u
3. $v.d < u.d < u.f < v.f$ and u is a descendant of v.



White-path theorem

Vertex v is a descendant of u if and only if at time $u.d$ there is a path from u to v consisting of only white vertices (except for u, which was just colored gray)



TOPOLOGICAL SORT

Application of DFS

Lecture 15, 09.04.2024

Topological sort

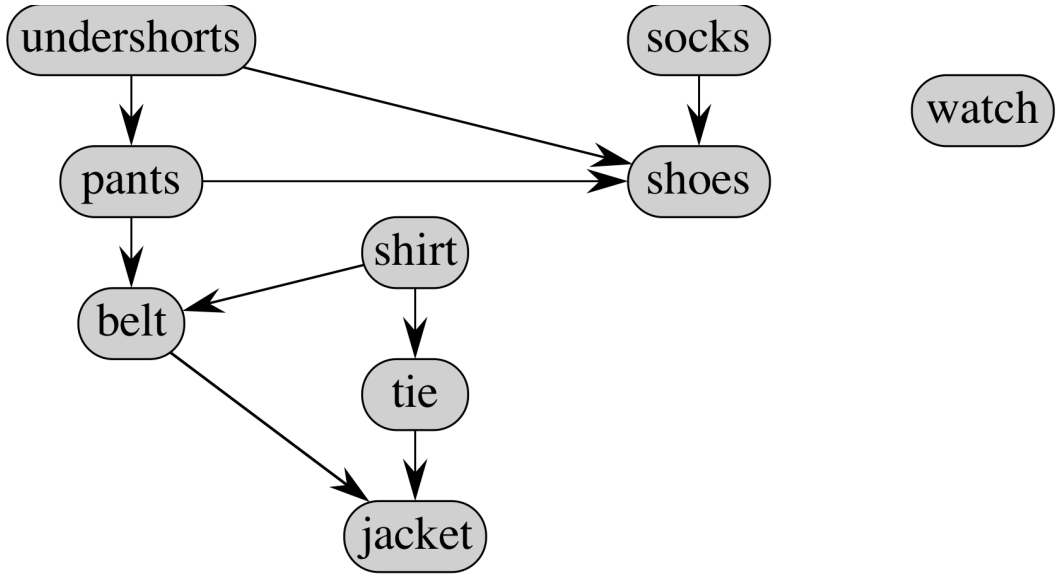
Definition

INPUT: A directed acyclic graph (DAG) $G = (V, E)$

OUTPUT: a linear ordering of vertices such that if $(u, v) \in E$, then u appears somewhere before v

Example

Getting dressed in the morning:



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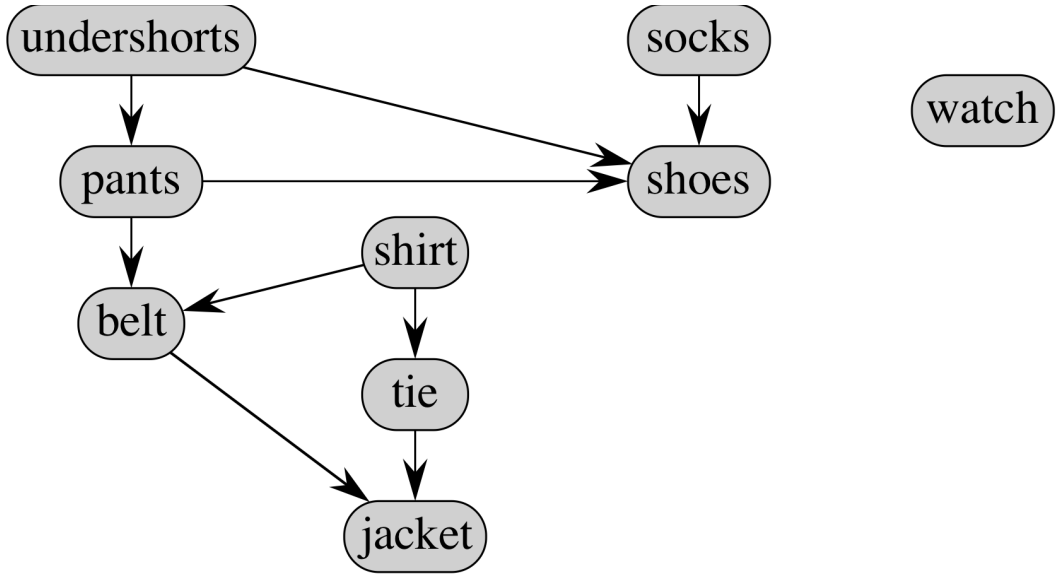
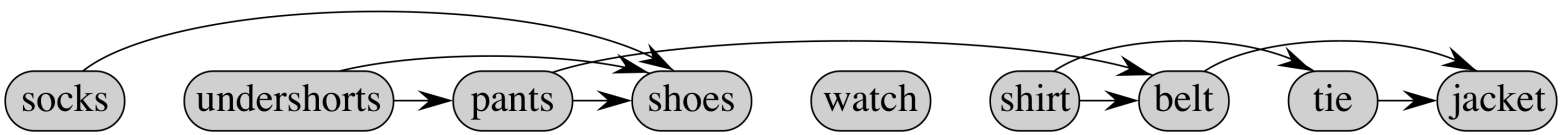
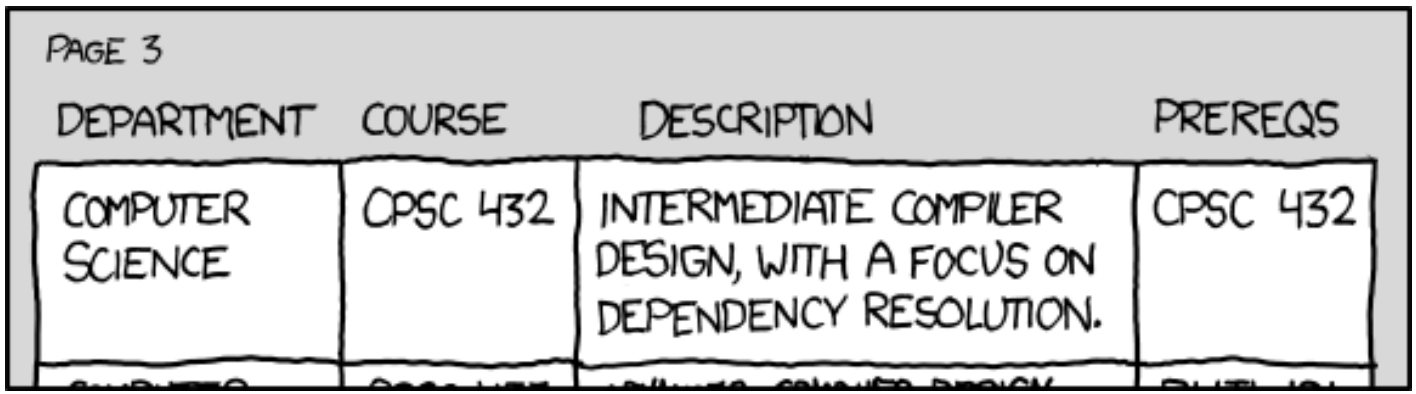


in which order?



First: when is a directed graph acyclic?



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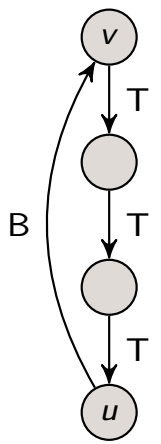
Suppose there is a back edge (u, v). Then v is ancestor of u in depth-first forest. Therefore there is a path from v to u, which creates a cycle.



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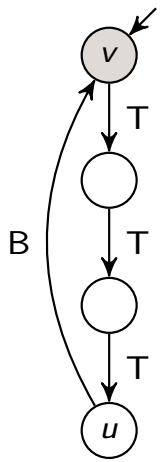
Proof. Second show that cycle implies back-edge
Let v be the first vertex discovered in the cycle C and let (u, v) be the preceding edge in C. At time $v.d$ vertices in C form a white-path from v to u and hence u is a descendant of v.



Algorithm for topological sort

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TOPOLOGICAL-SORT(G):
1. Call $DFS(G)$ to compute finishing times $v.f$ for all $v \in G.V$
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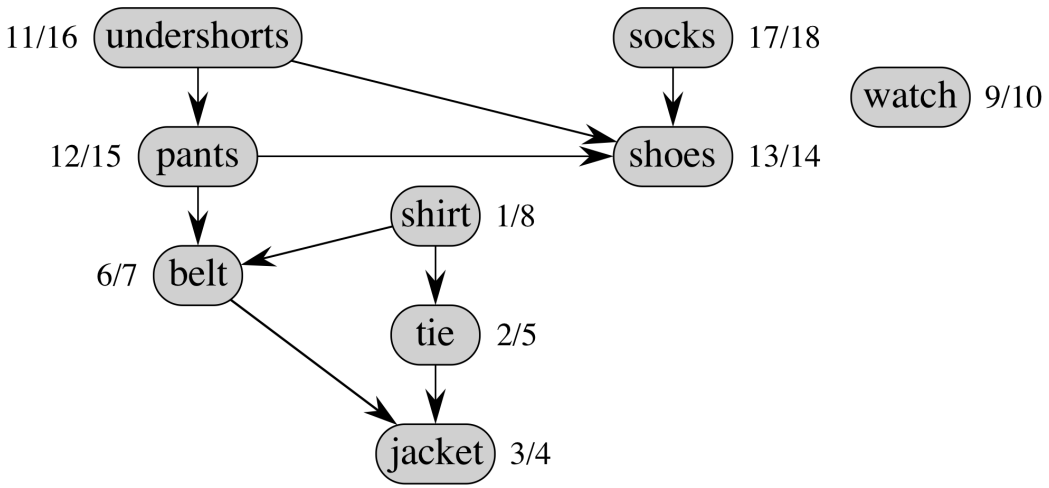

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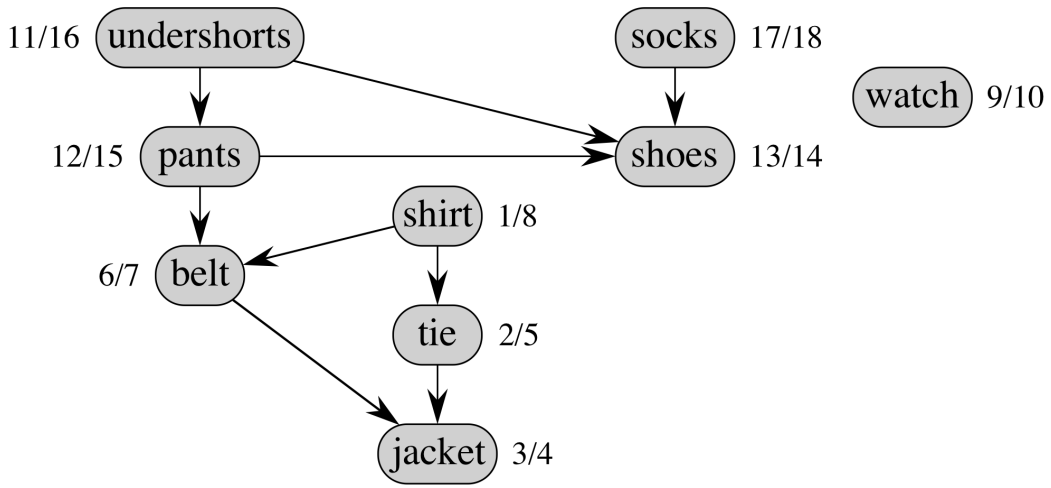

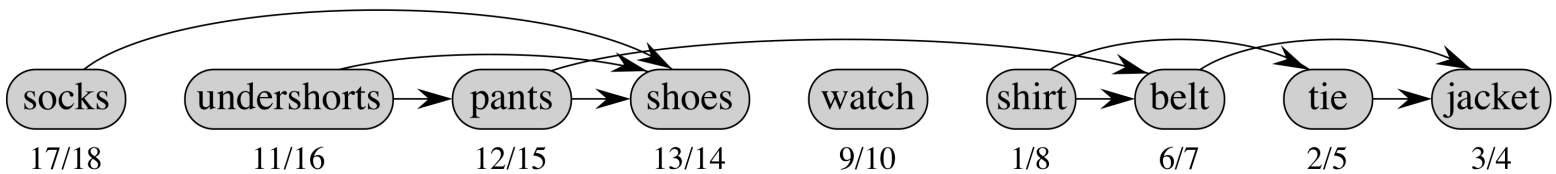

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Time: $\Theta(V + E)$ (same as DFS)

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STRONGLY CONNECTED COMPONENTS
(A magic algorithm)

Lecture 15, 09.04.2024

What is a Strongly Connected Component?

Definition: A strongly connected component (SCC) of a directed graph $G = (V, E)$ is a maximal set of vertices $C \subseteq V$ such that for all $u, v \in C$, both $u \rightsquigarrow v$ and $v \rightsquigarrow u$.

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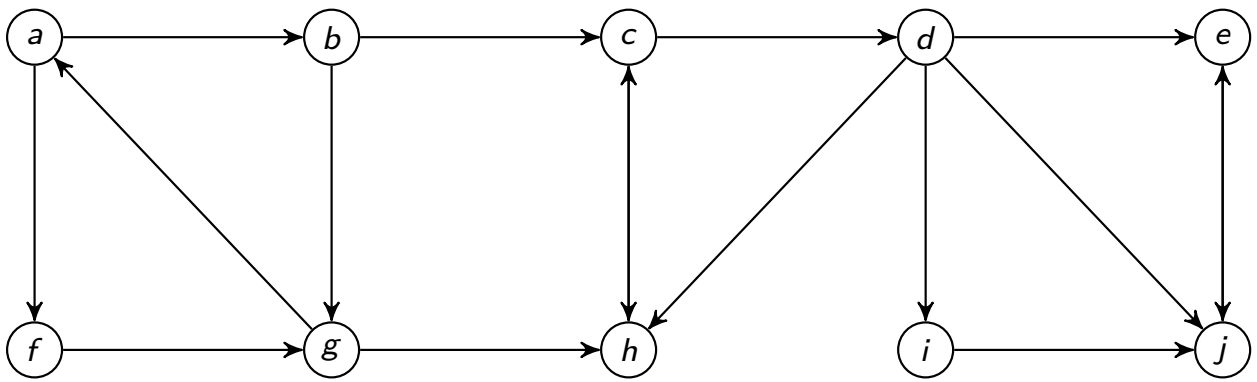

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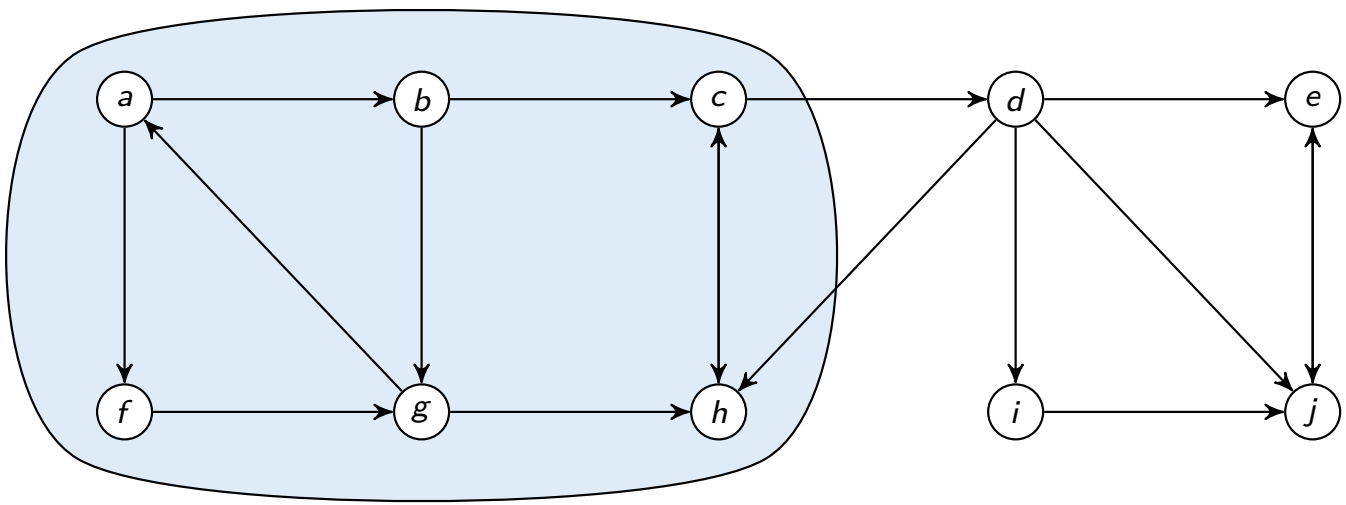

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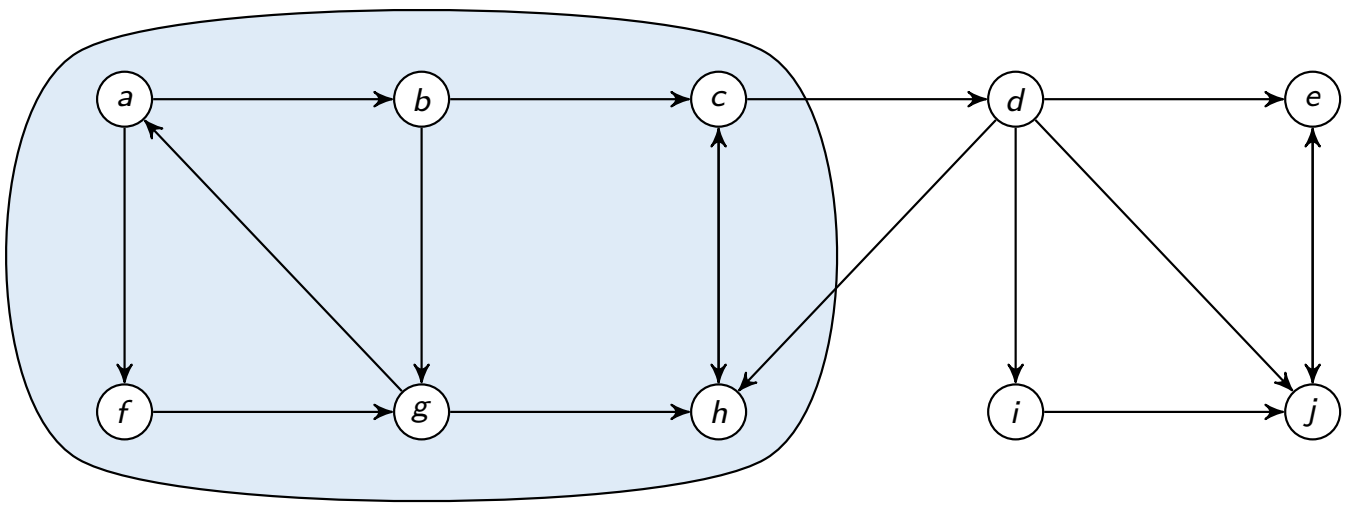

Is this a SCC? NO, because e.g. $c \not\rightsquigarrow b$

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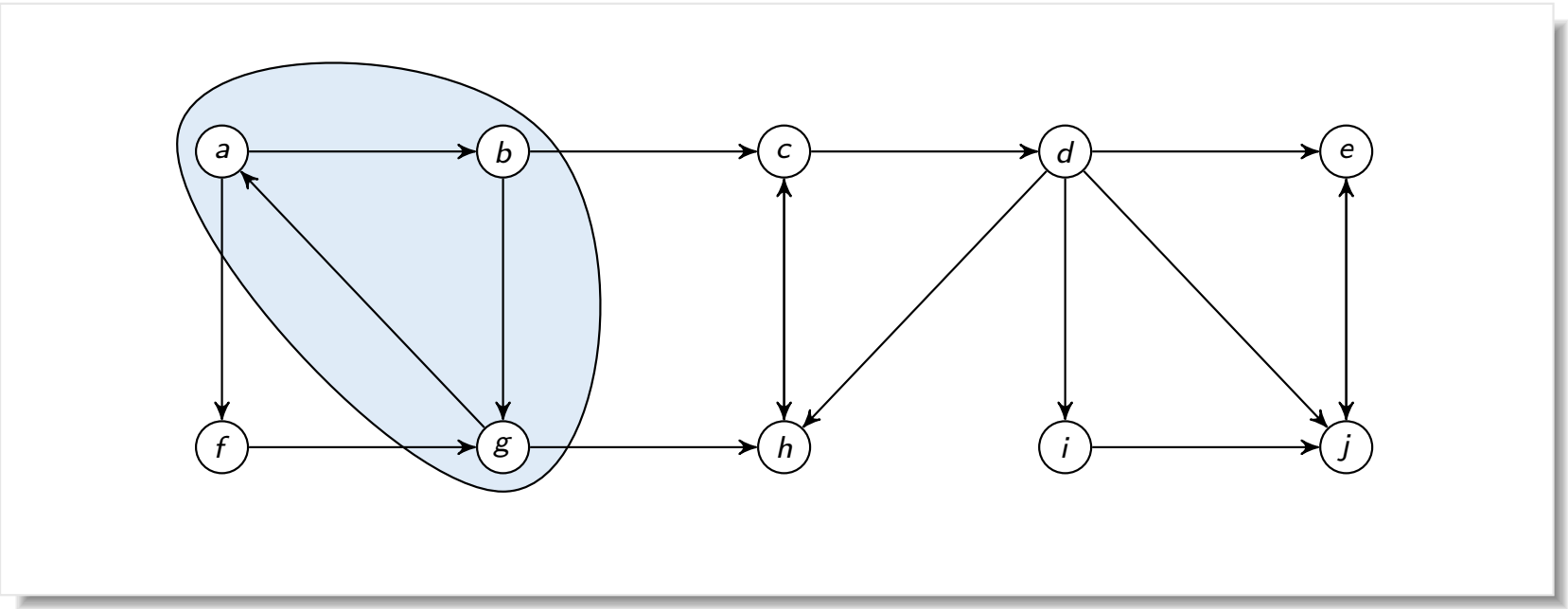

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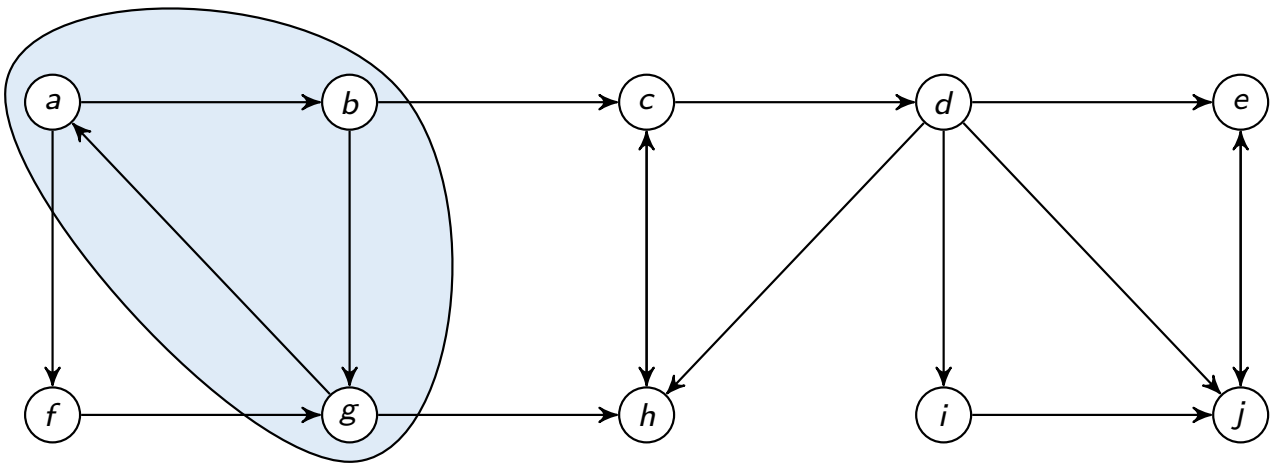

Is this a SCC? NO, because not maximal

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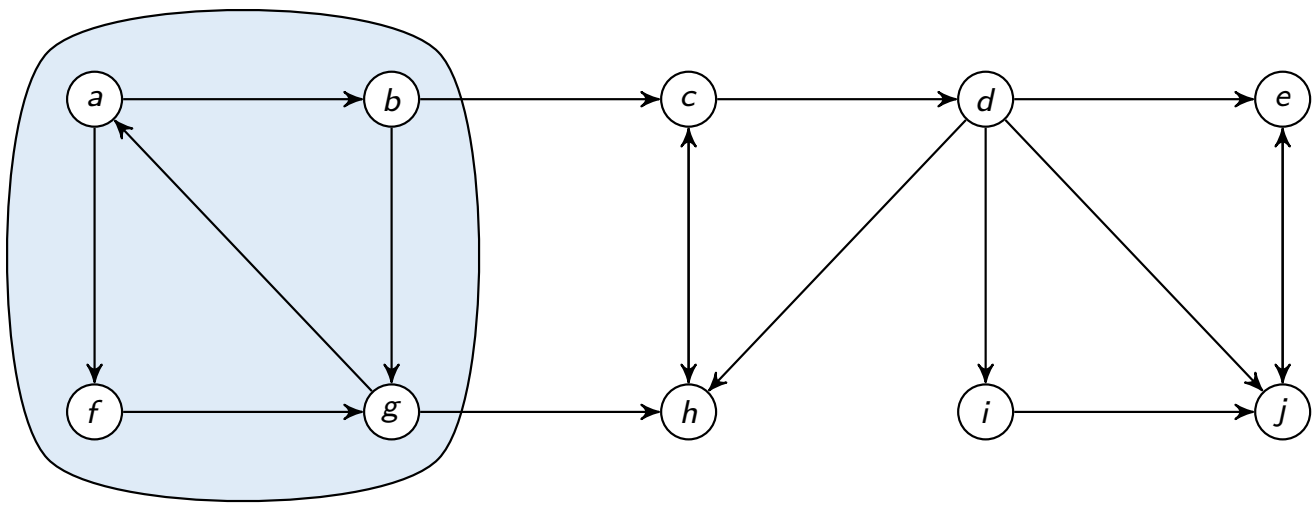

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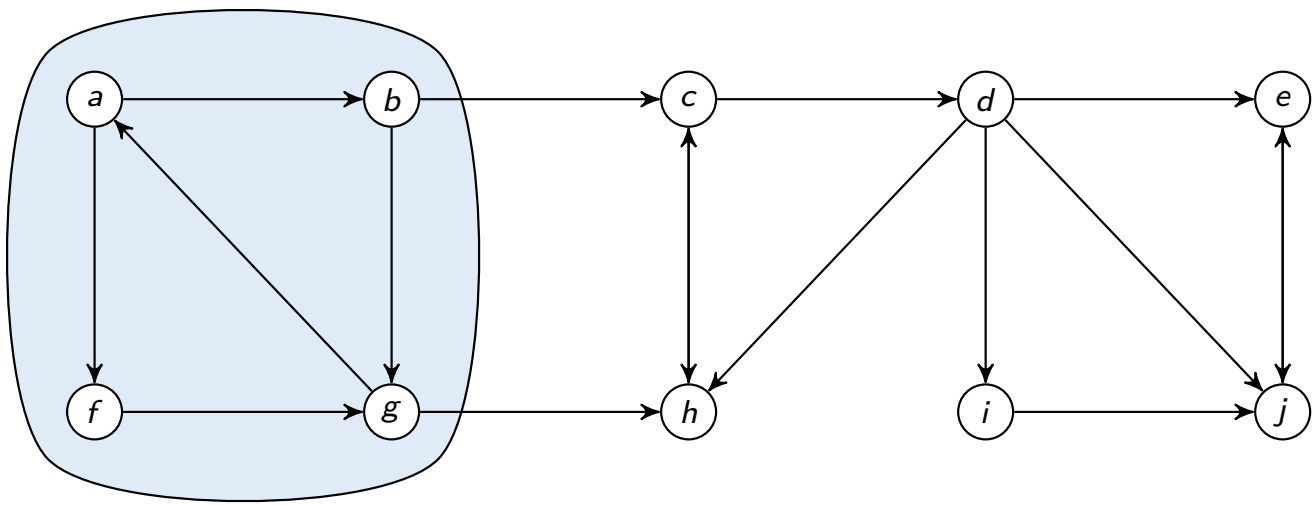

Is this a SCC? YES!

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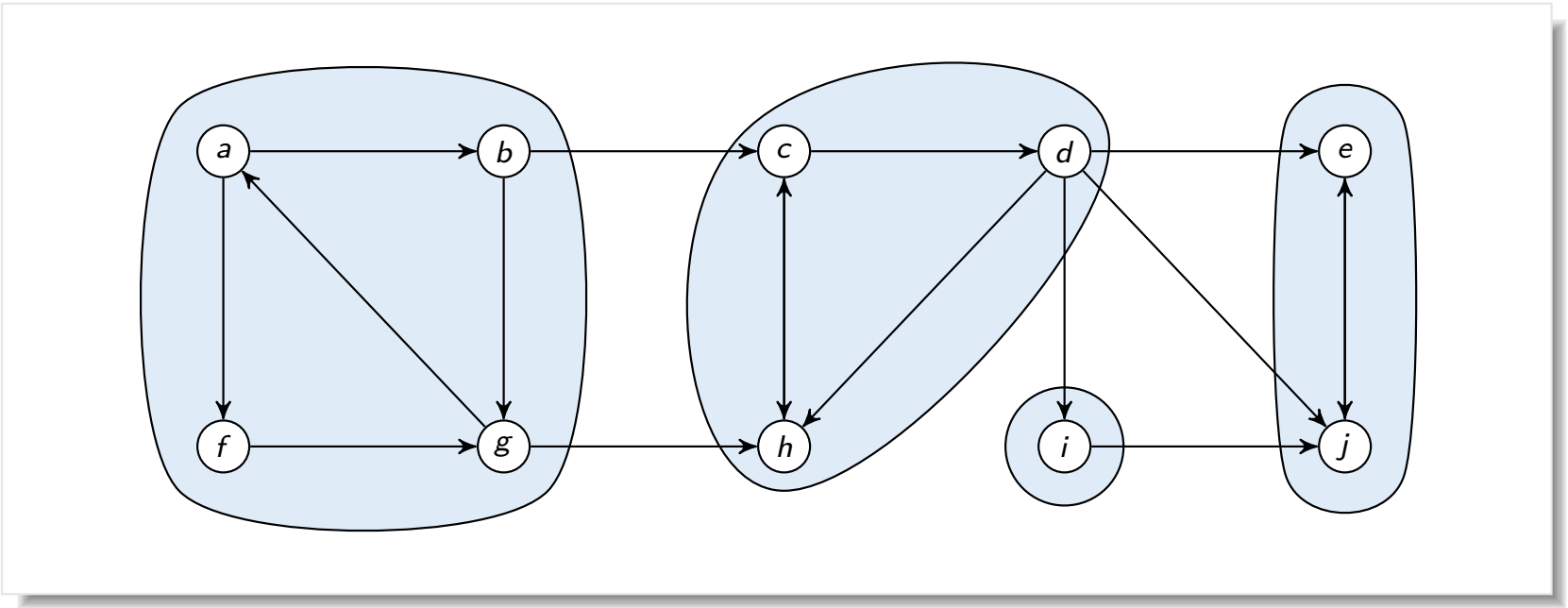

A depiction of all SCCs of the graph

Component Graph

For a digraph $G = (V, E)$, its component graph $G^{\mathrm{SCC}} = (V^{\mathrm{SCC}}, E^{\mathrm{SCC}})$ is defined by:

- V^{SCC} has a vertex for each SCC in G;
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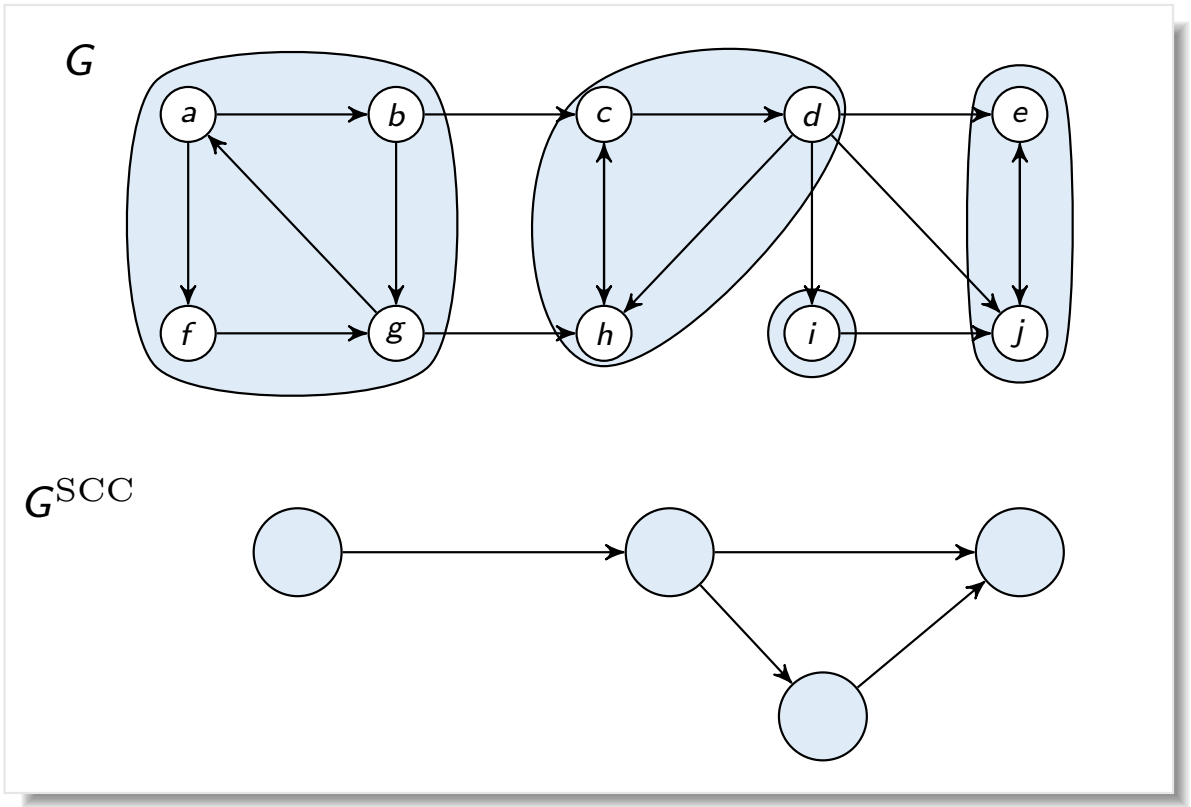

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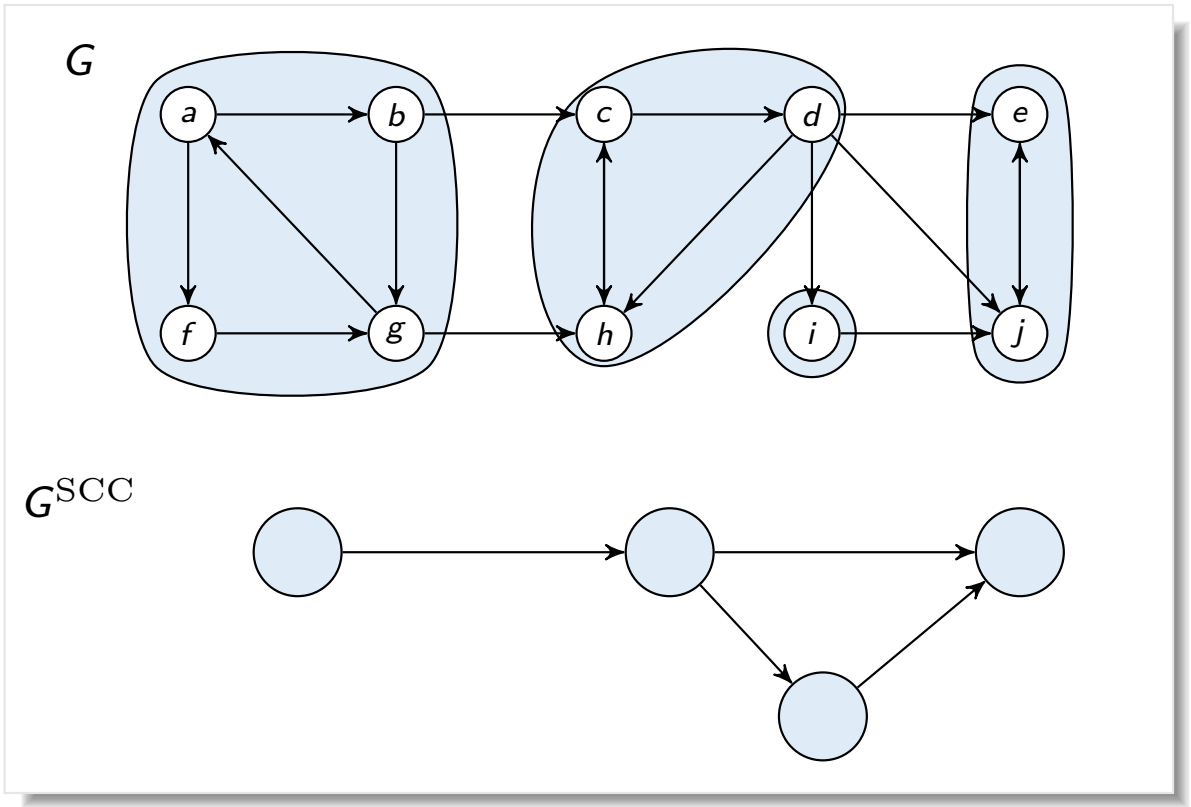

Lemma: G^{SCC} is a DAG.

Magic Algorithm

$\mathrm{SCC}(G)$:

1. Call $\mathrm{DFS}(G)$ to compute finishing times $u.f$ for all u.
2. Compute G^T
3. Call $\mathrm{DFS}(G^T)$ but in the main loop, consider vertices in order of decreasing $u.f$ (as computed in first DFS).
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Graph G^T is the transpose of G:

- $G^T = (V, E), E^T = \{(u, v) : (v, u) \in E\}$.
- G^T is G with all edges reversed.

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Observations:

- Can create G^T in $\Theta(V + E)$ time if using adjacency lists.
- G and G^T has the same SCCs.

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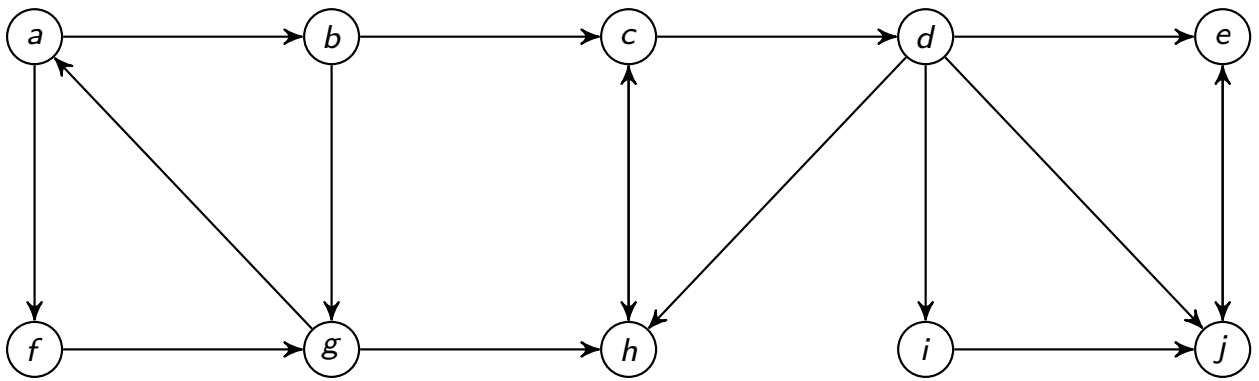

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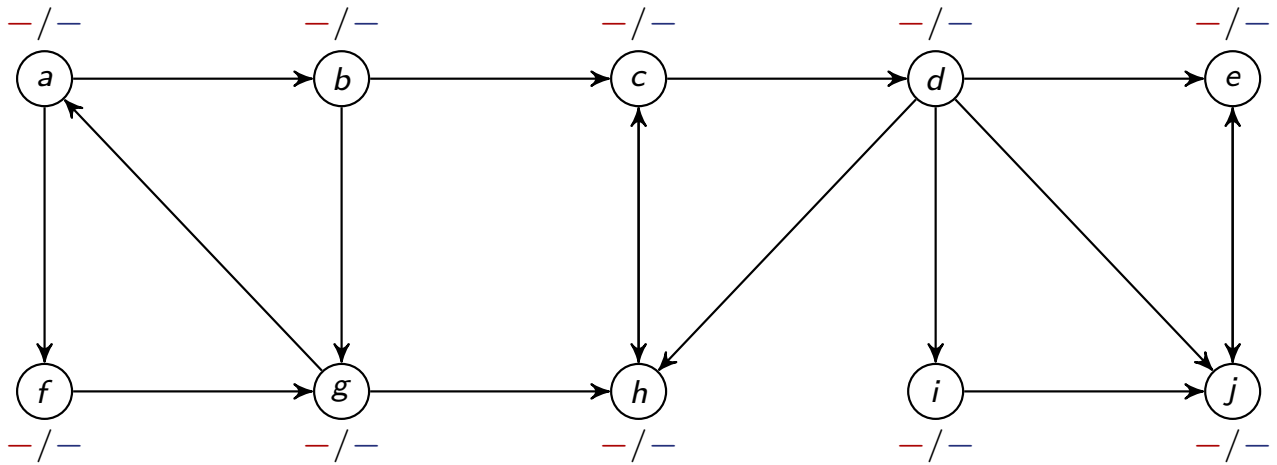

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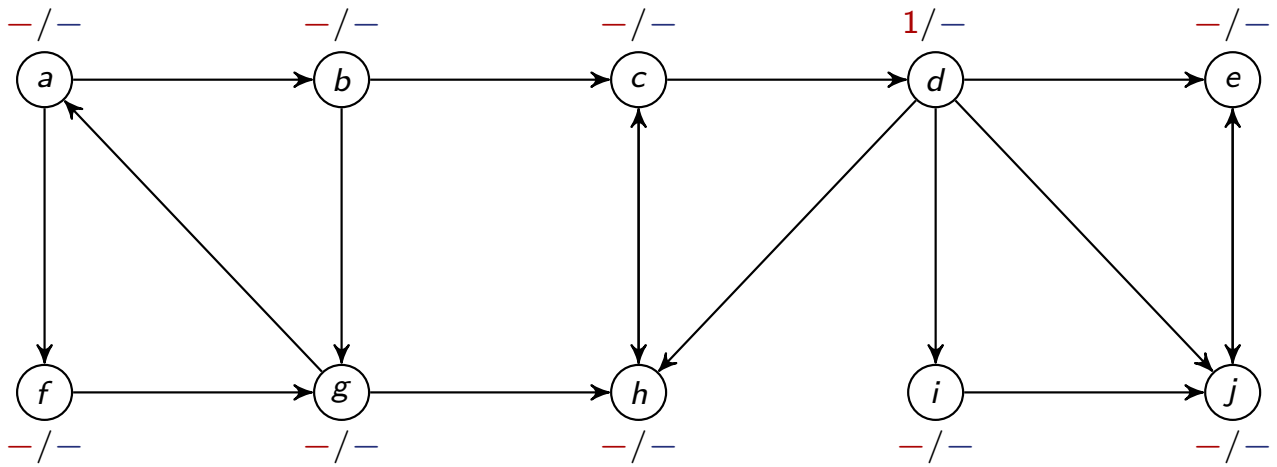

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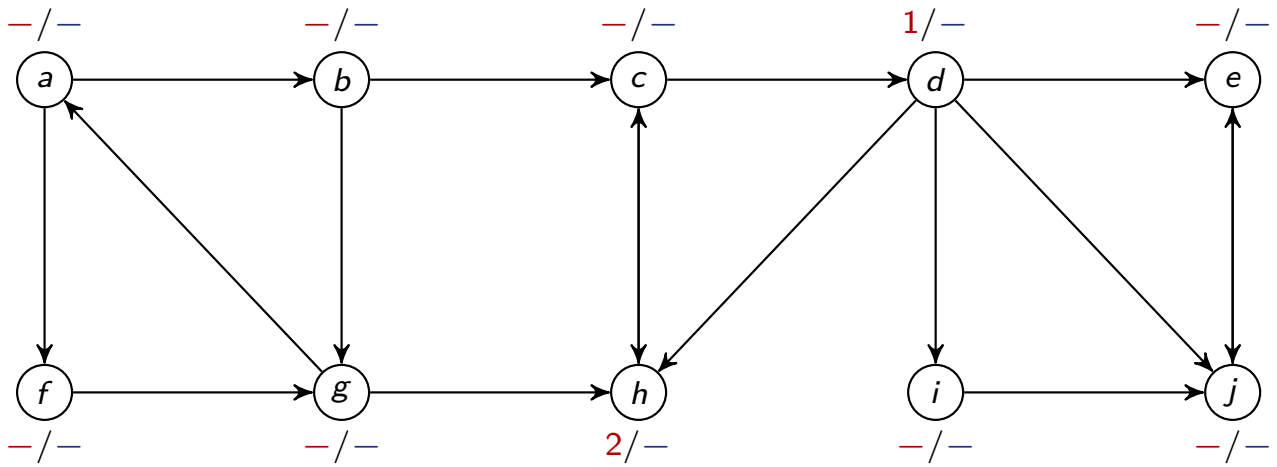

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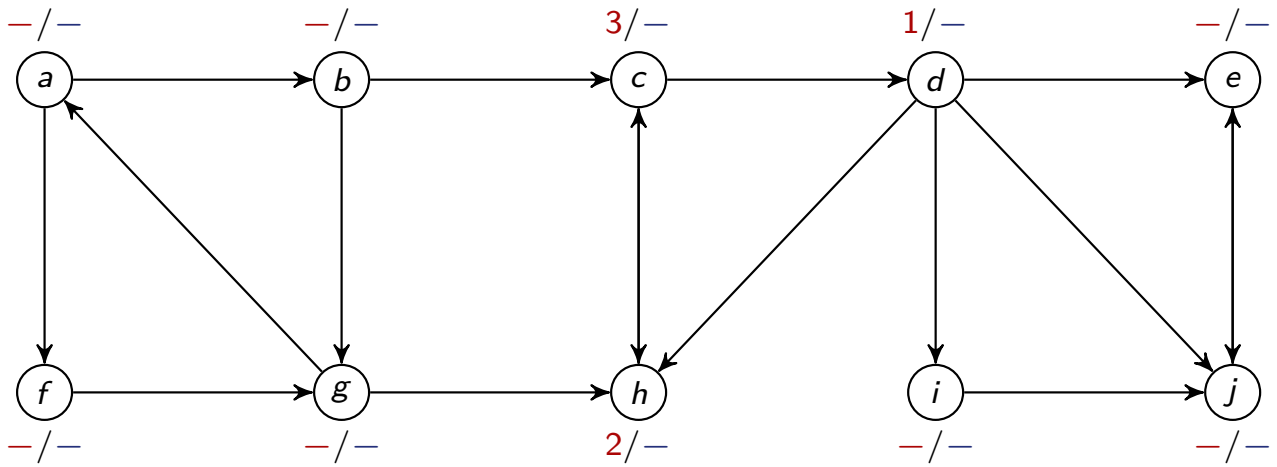

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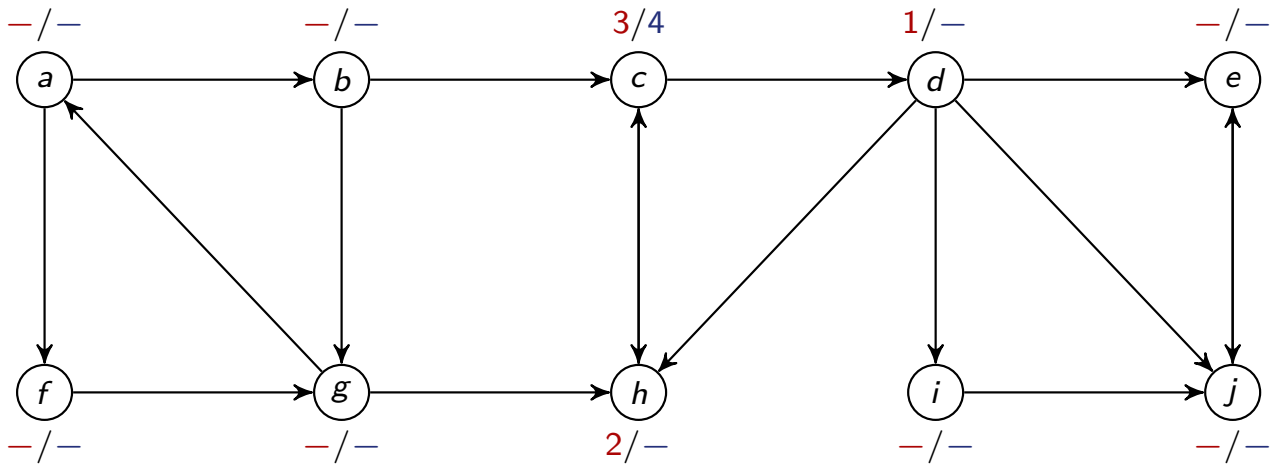

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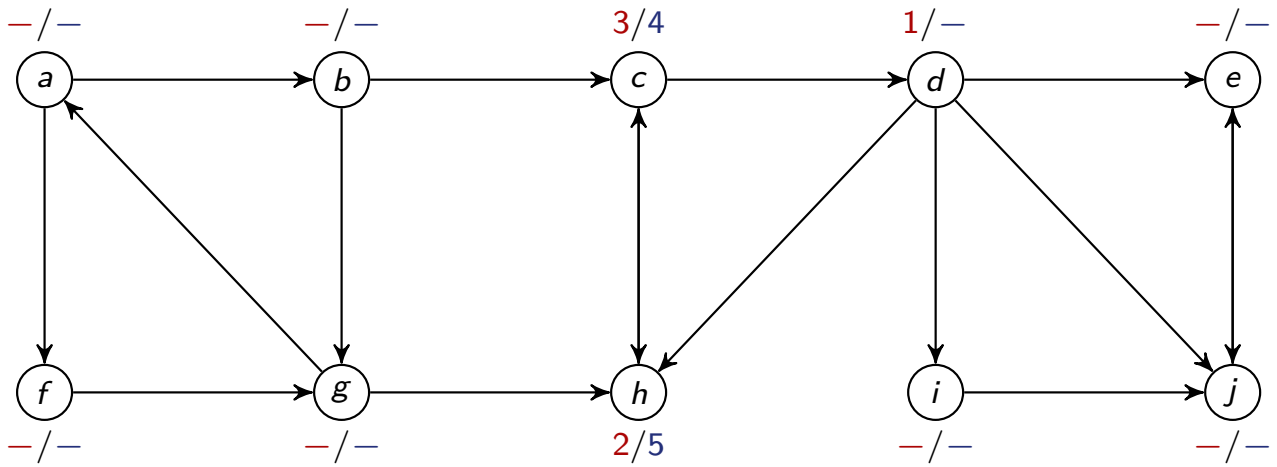

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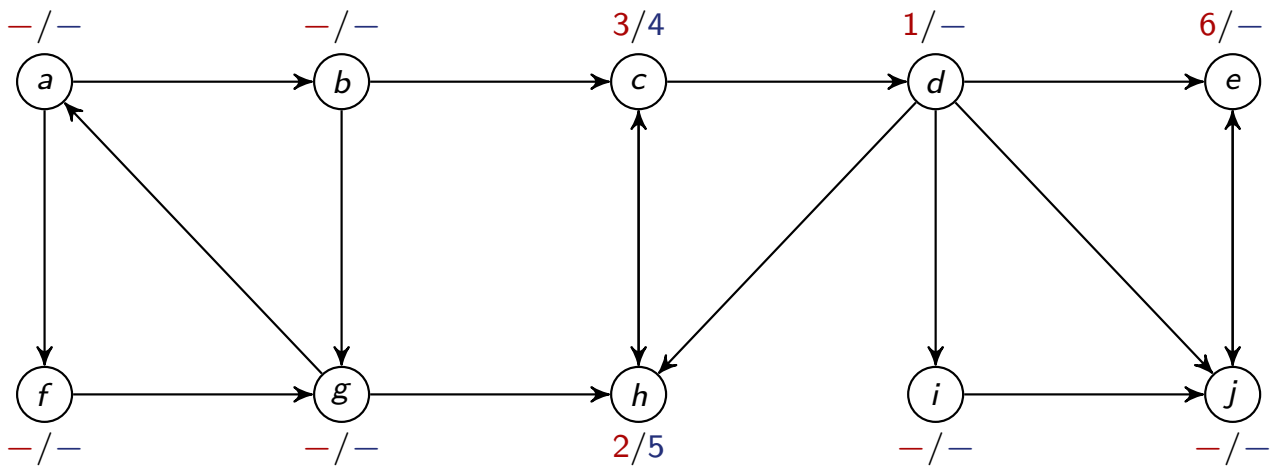

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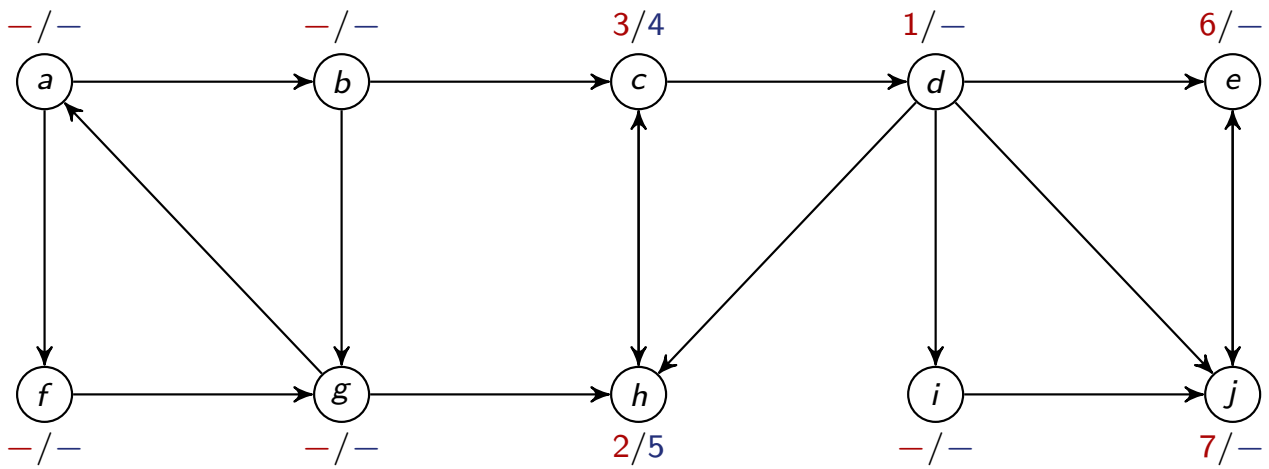

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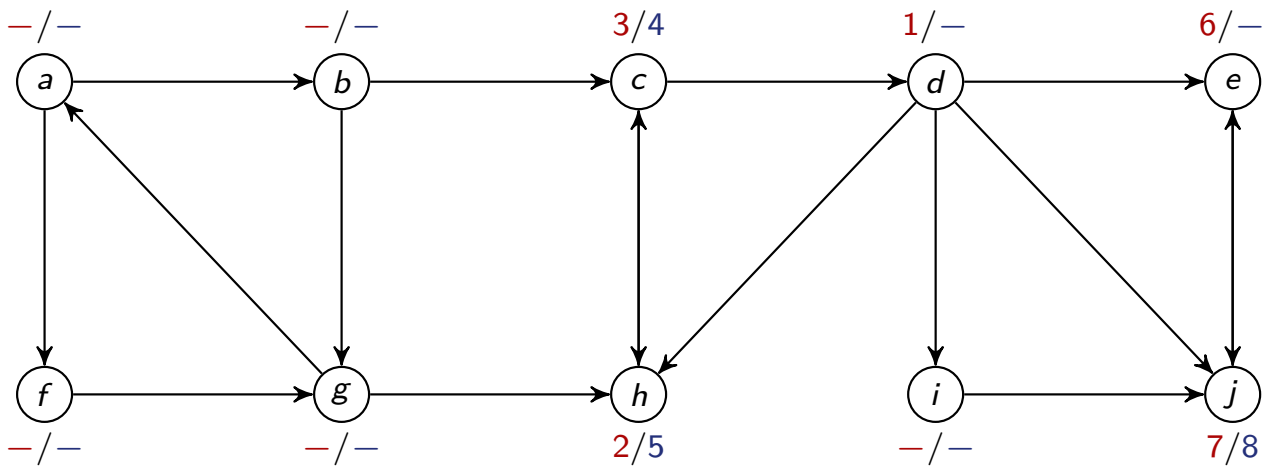

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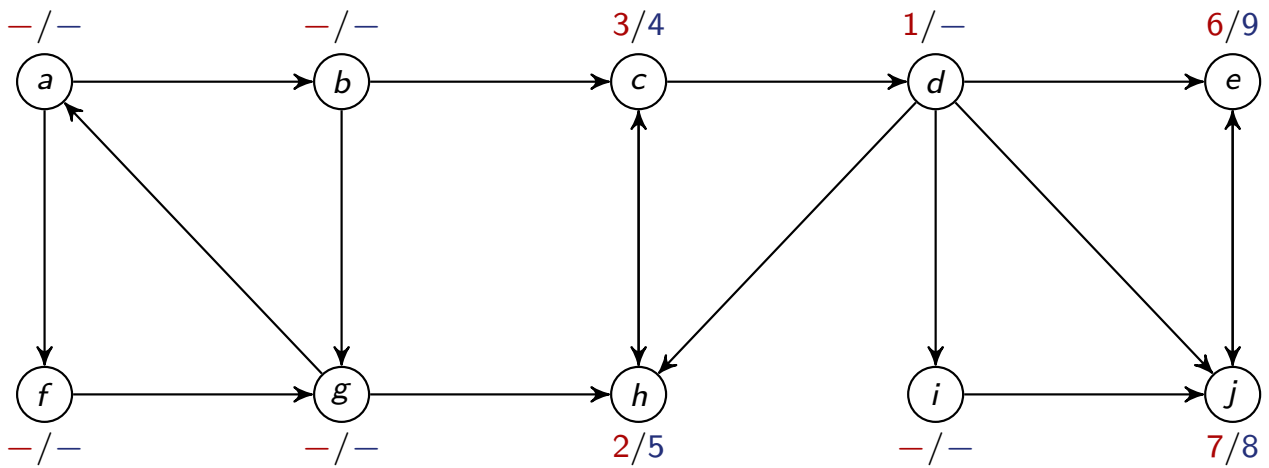

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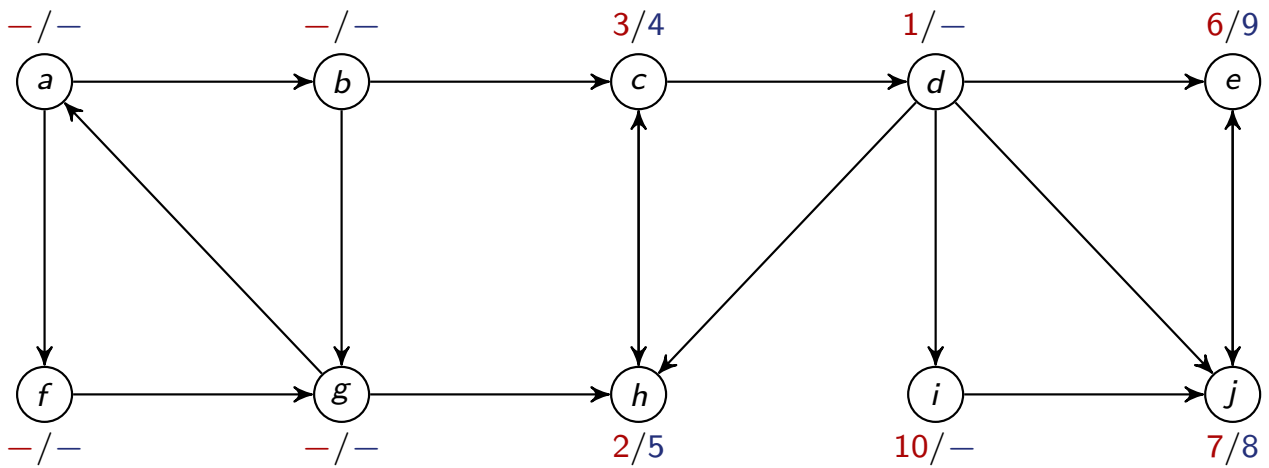

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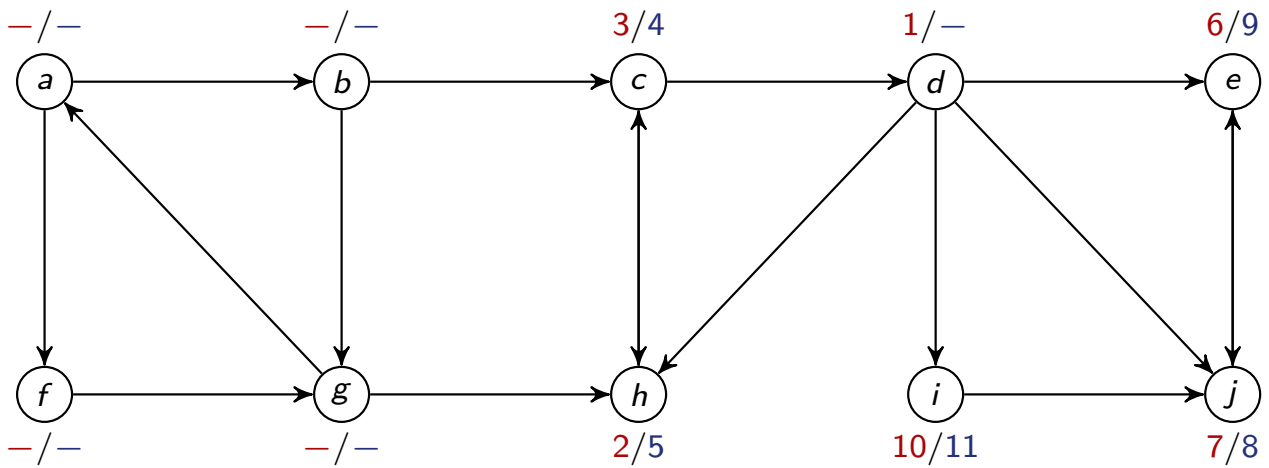

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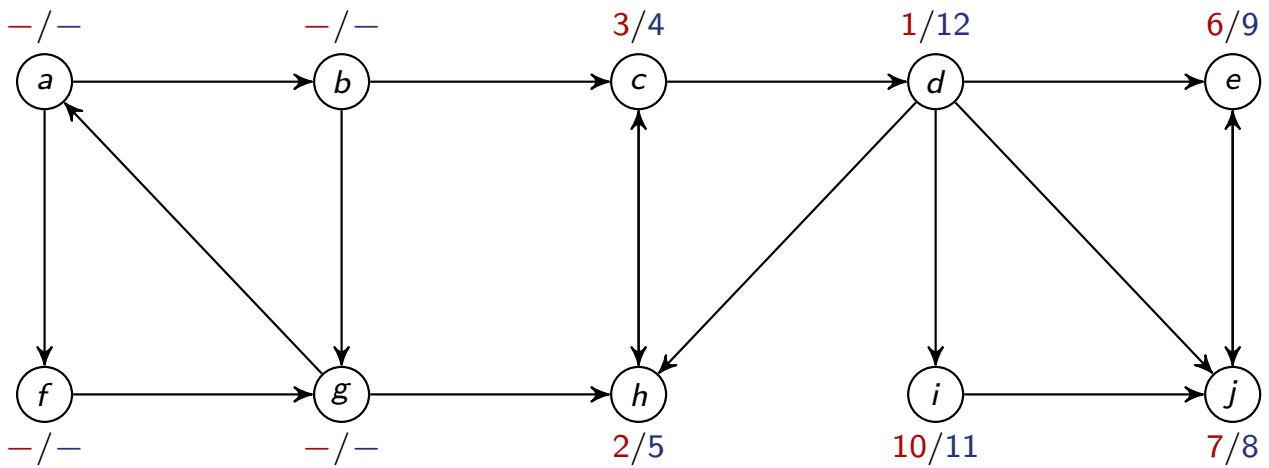

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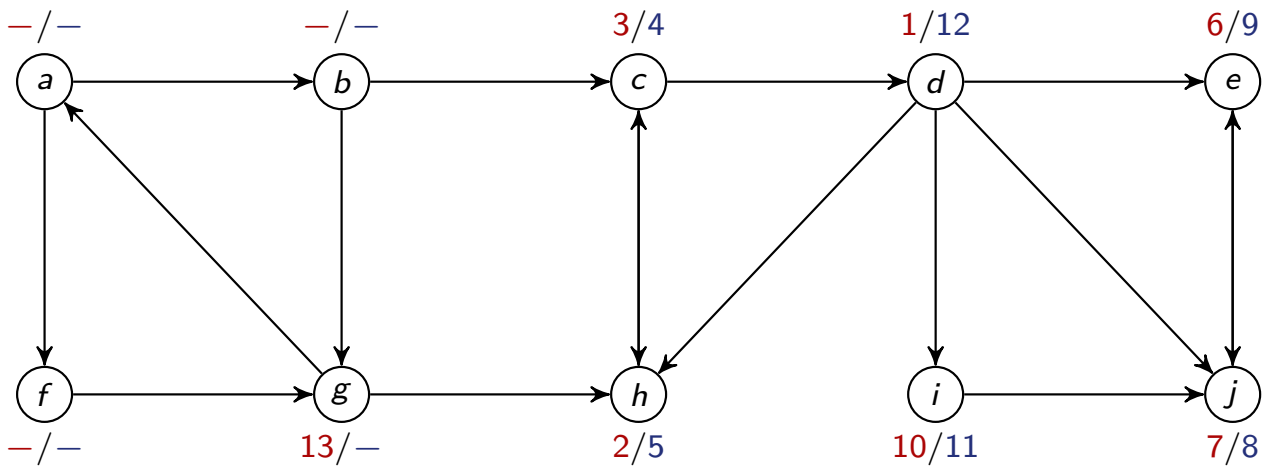

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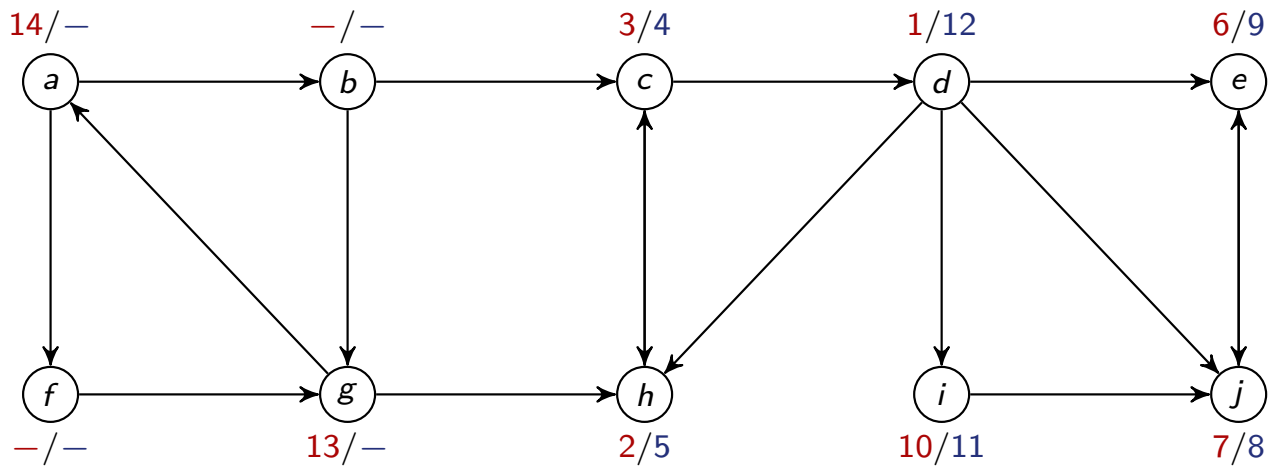

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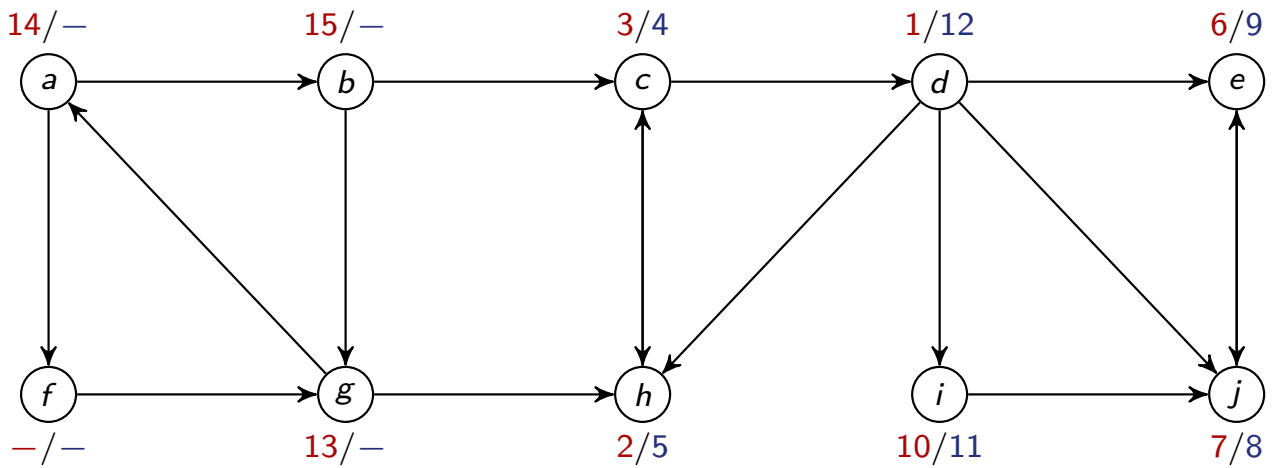

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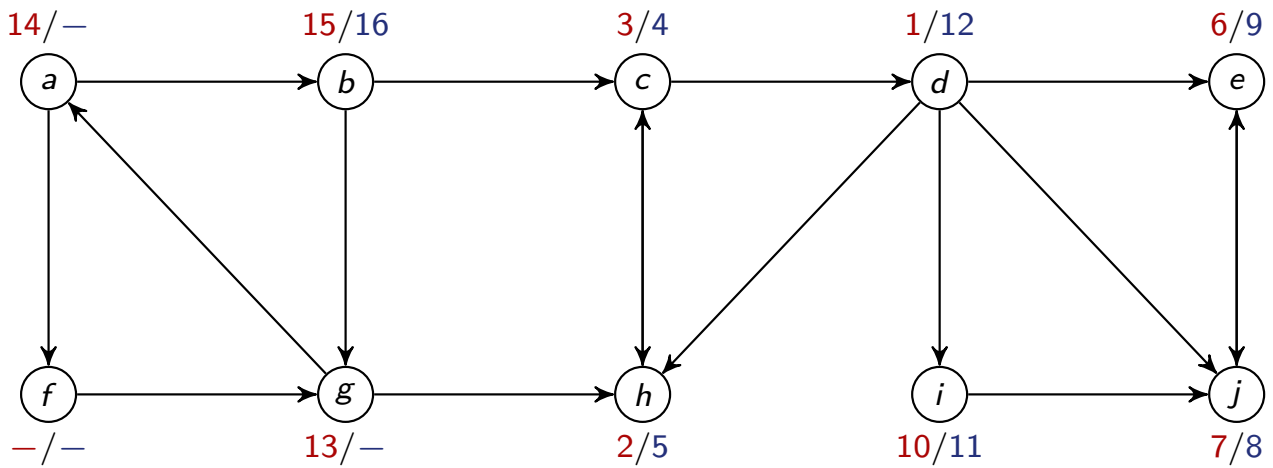

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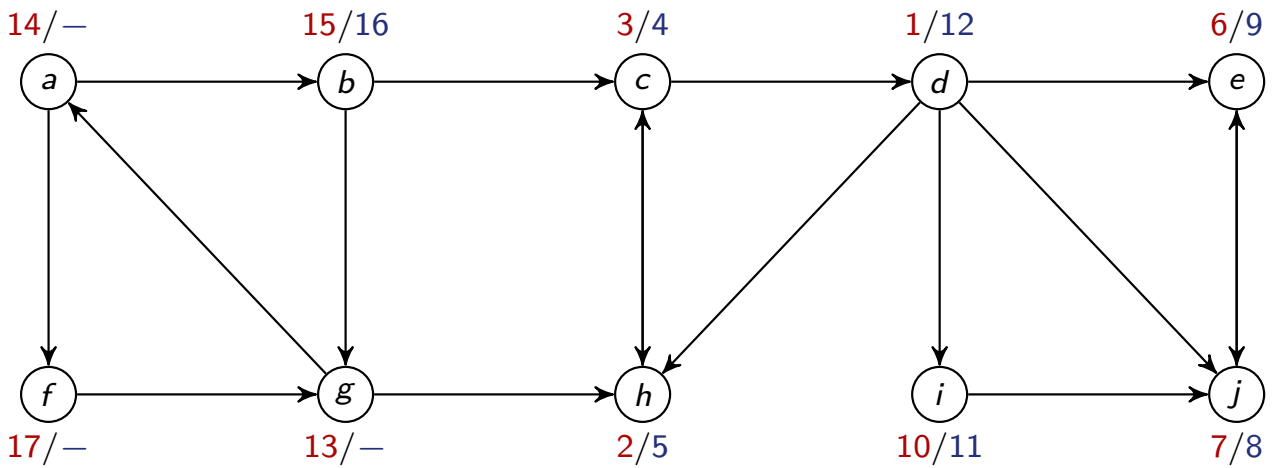

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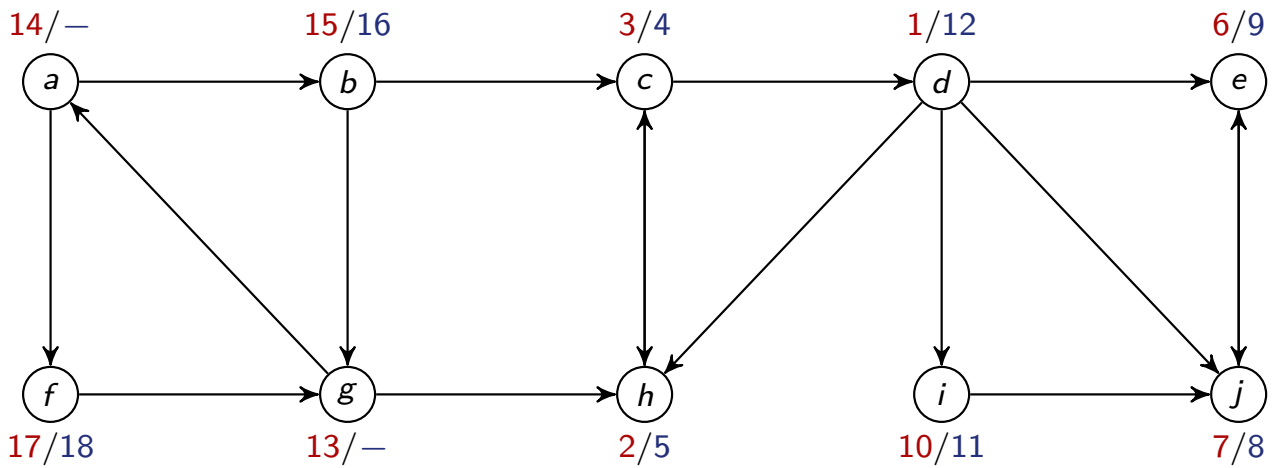

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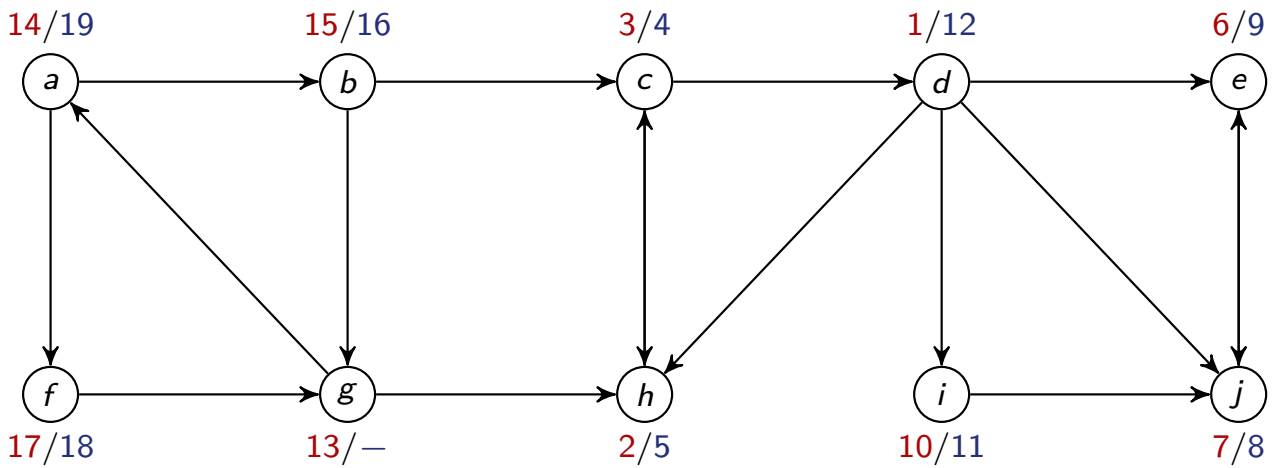

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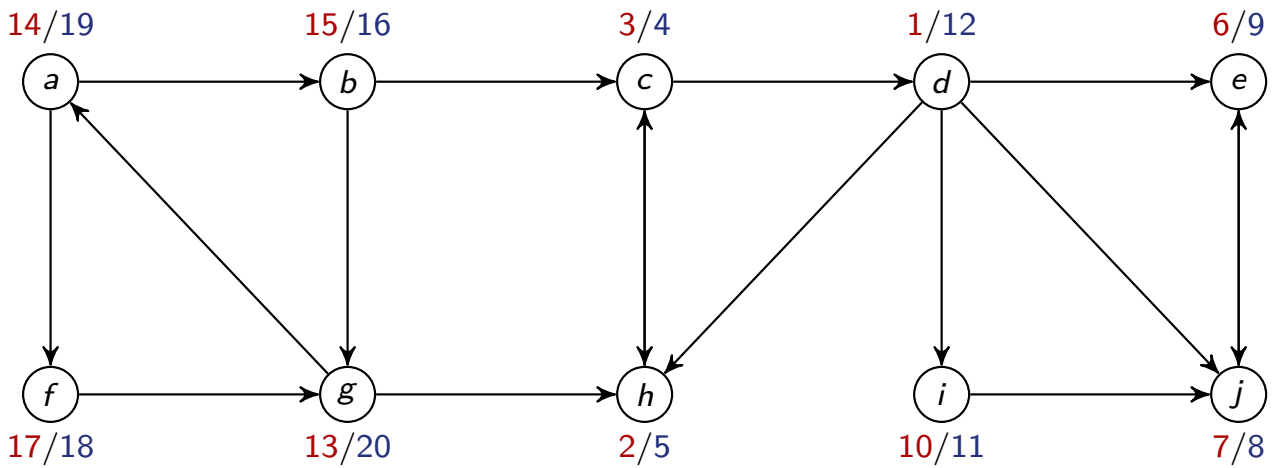

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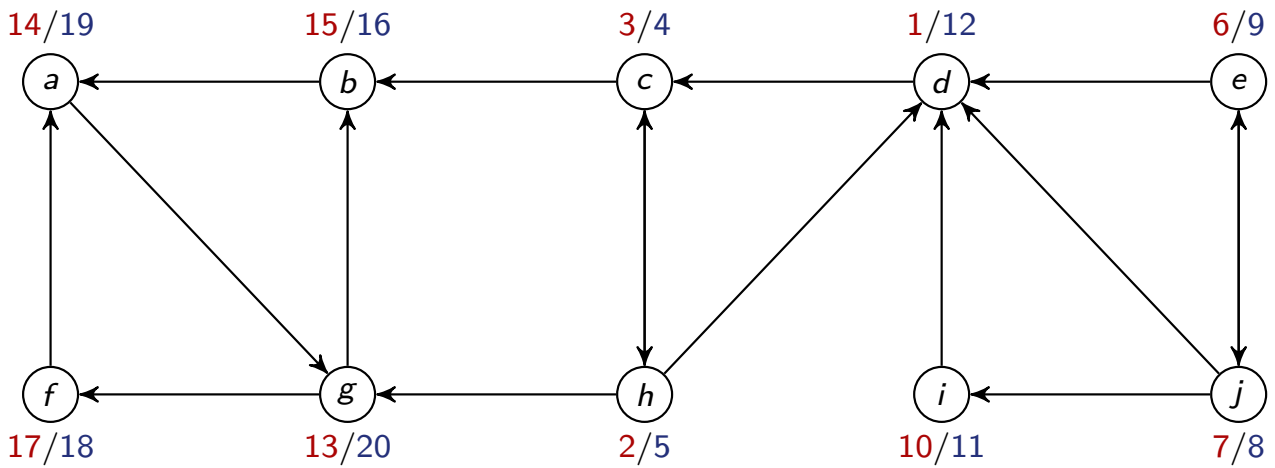

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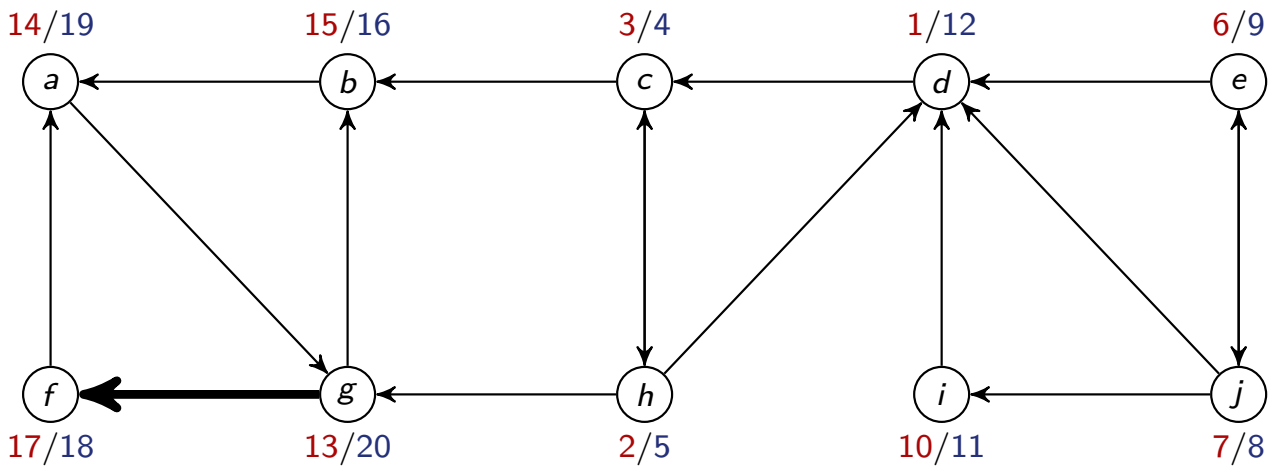

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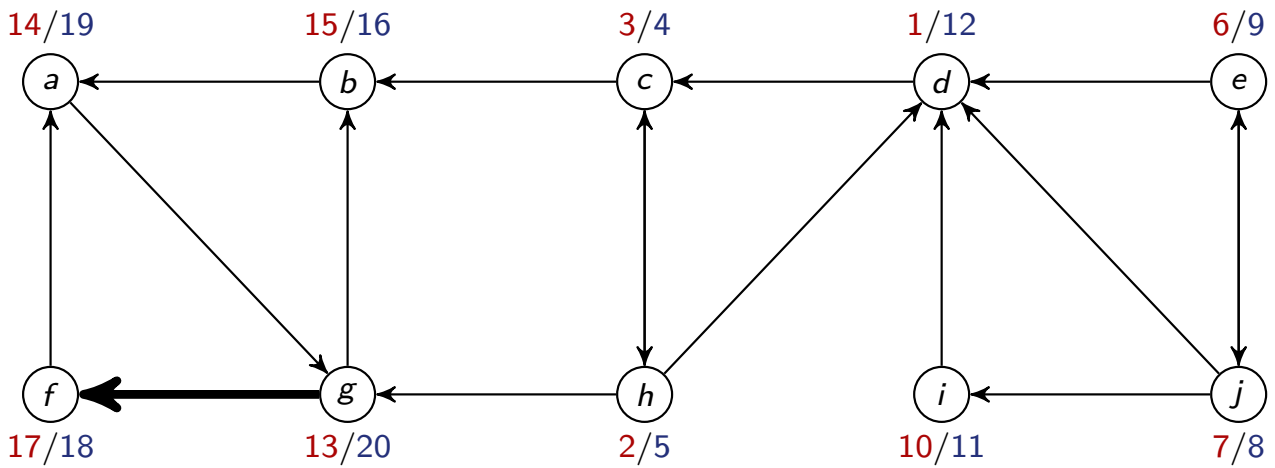

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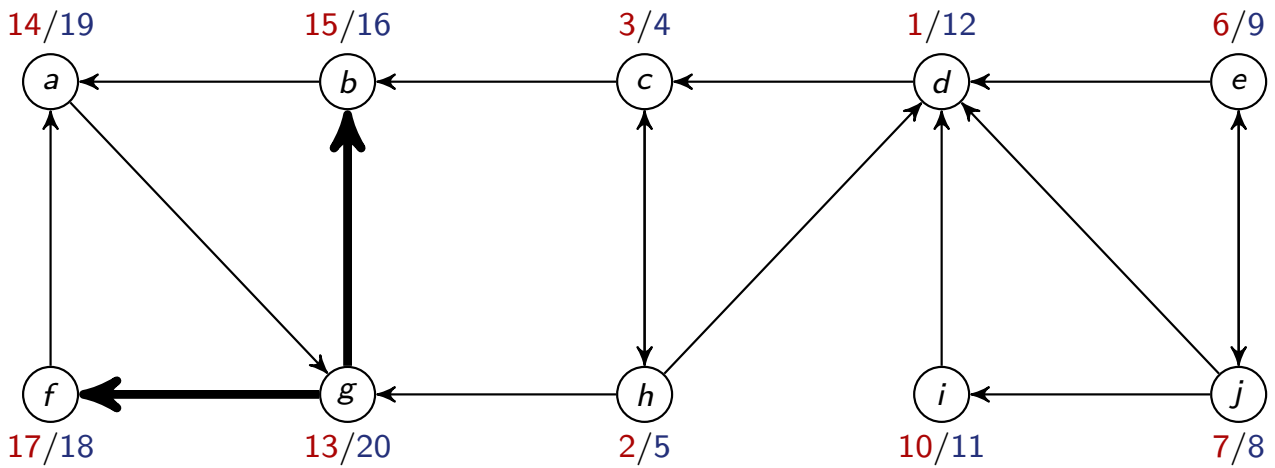

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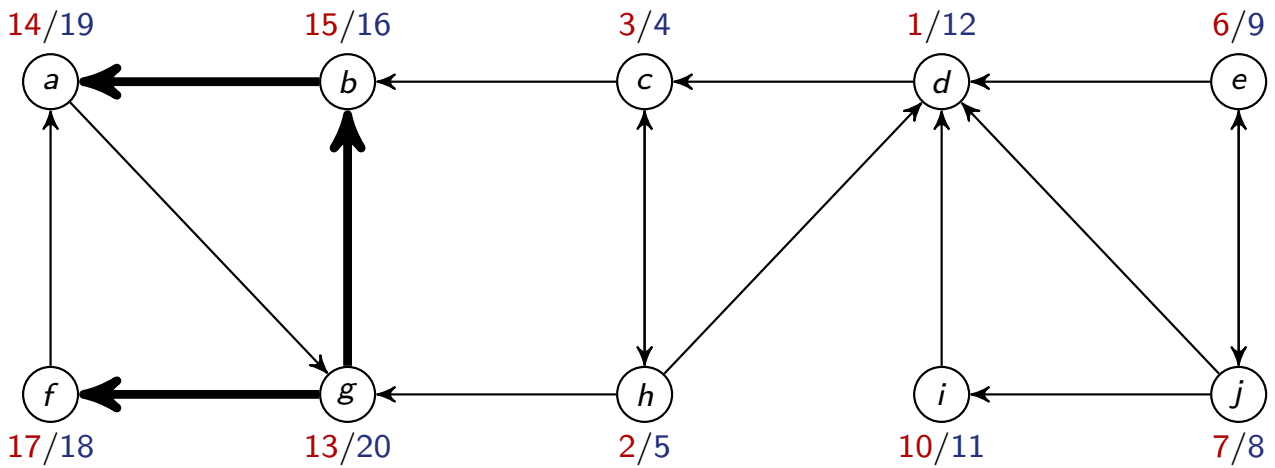

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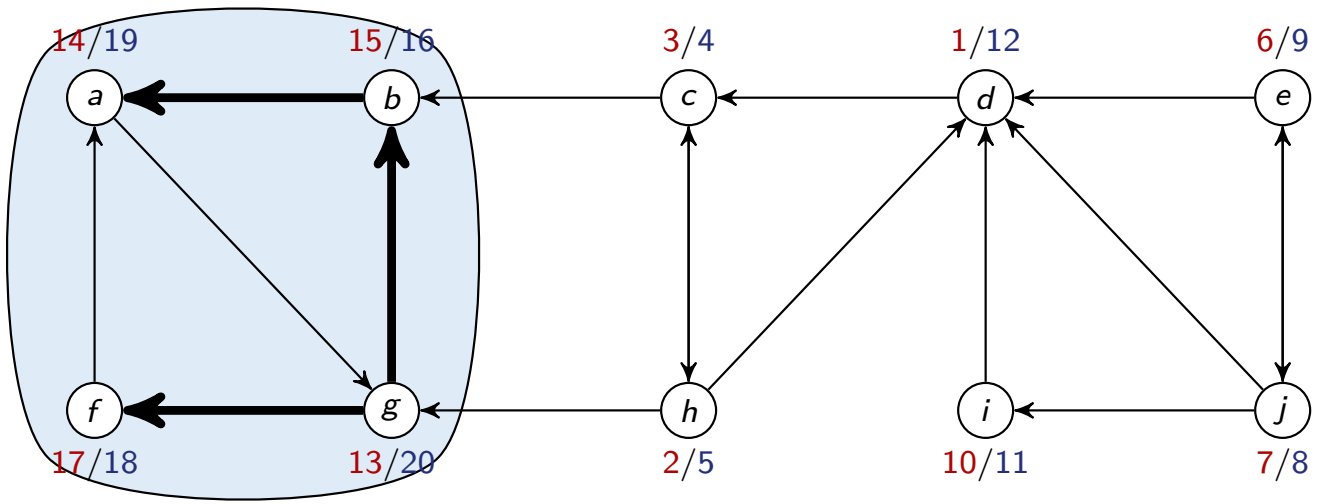

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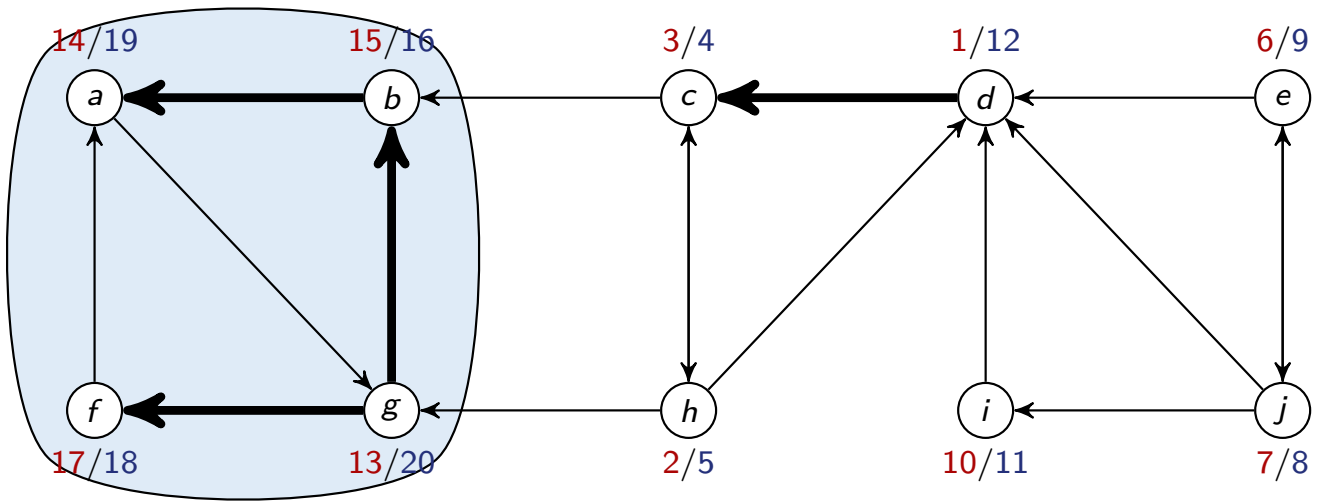

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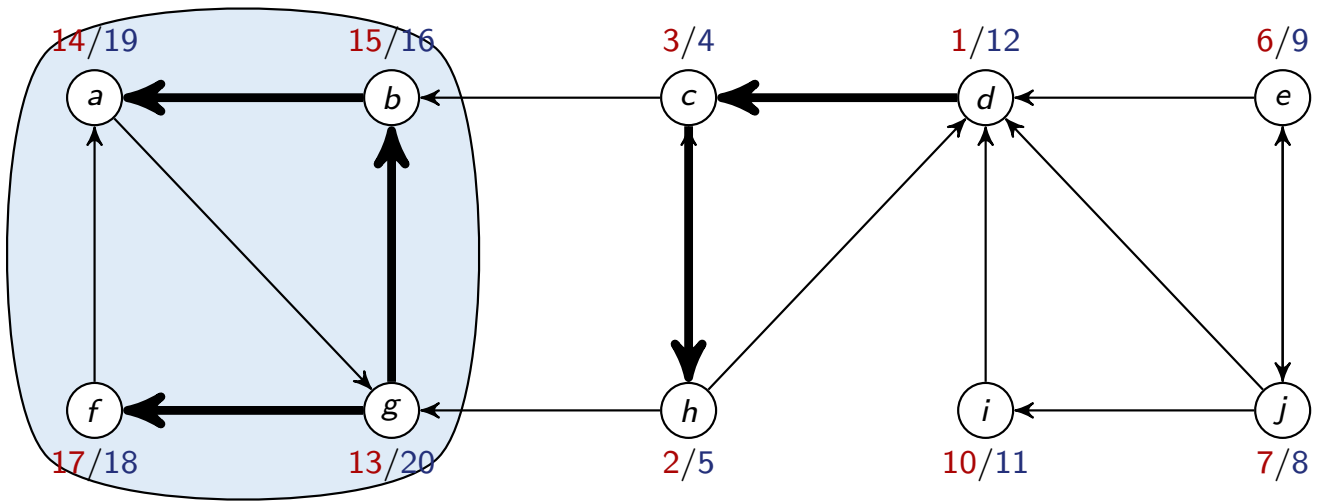

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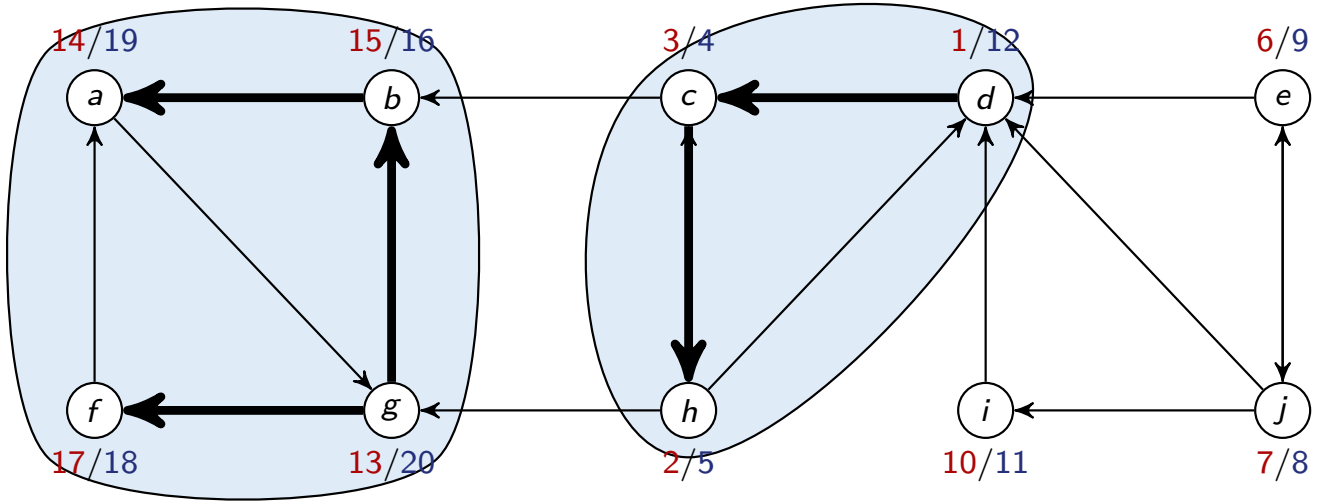

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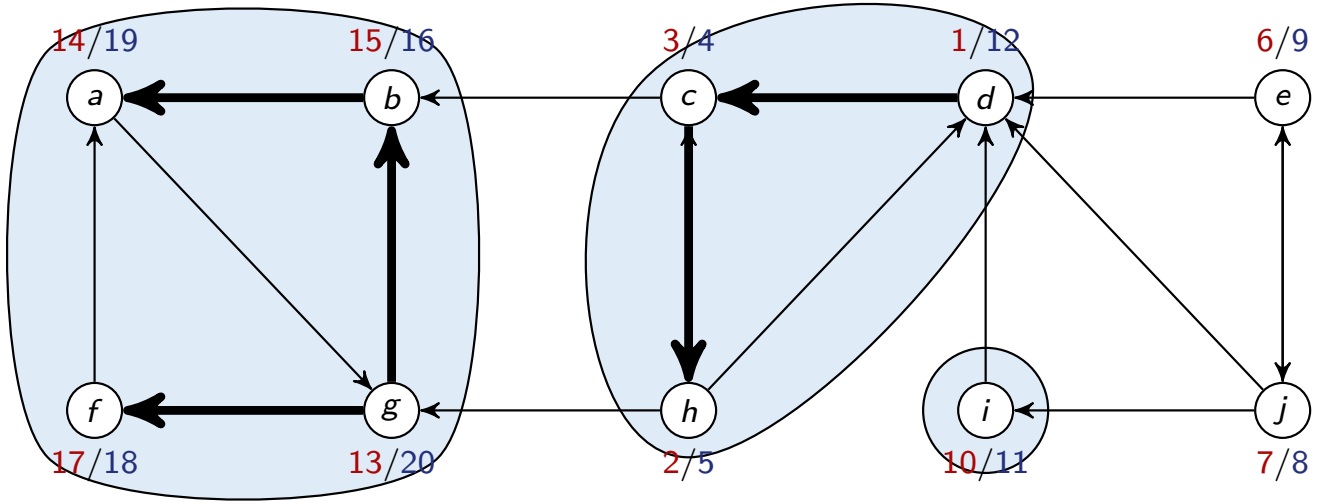

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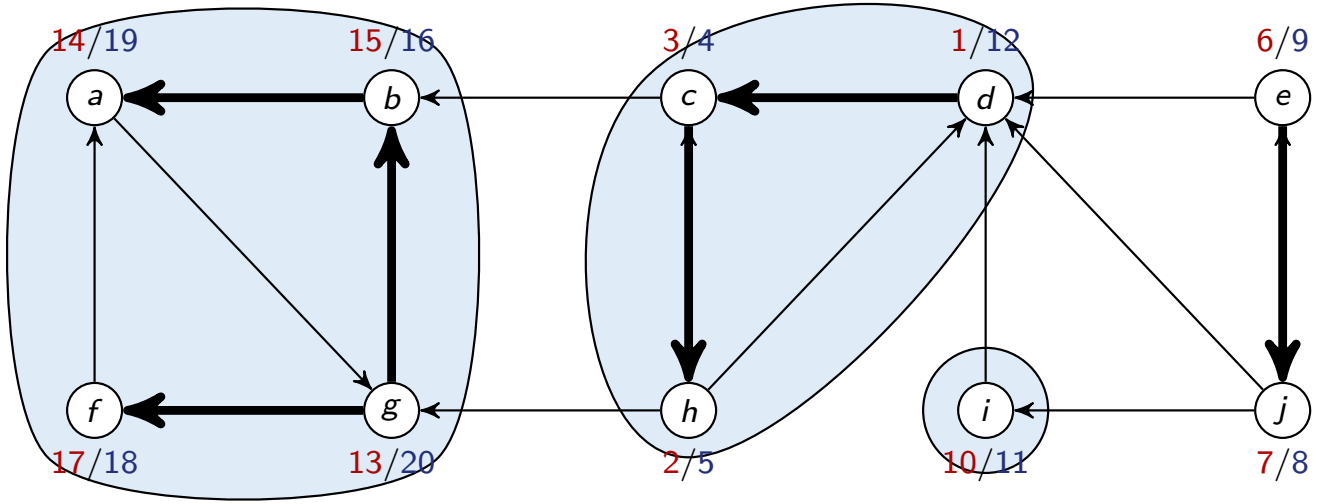

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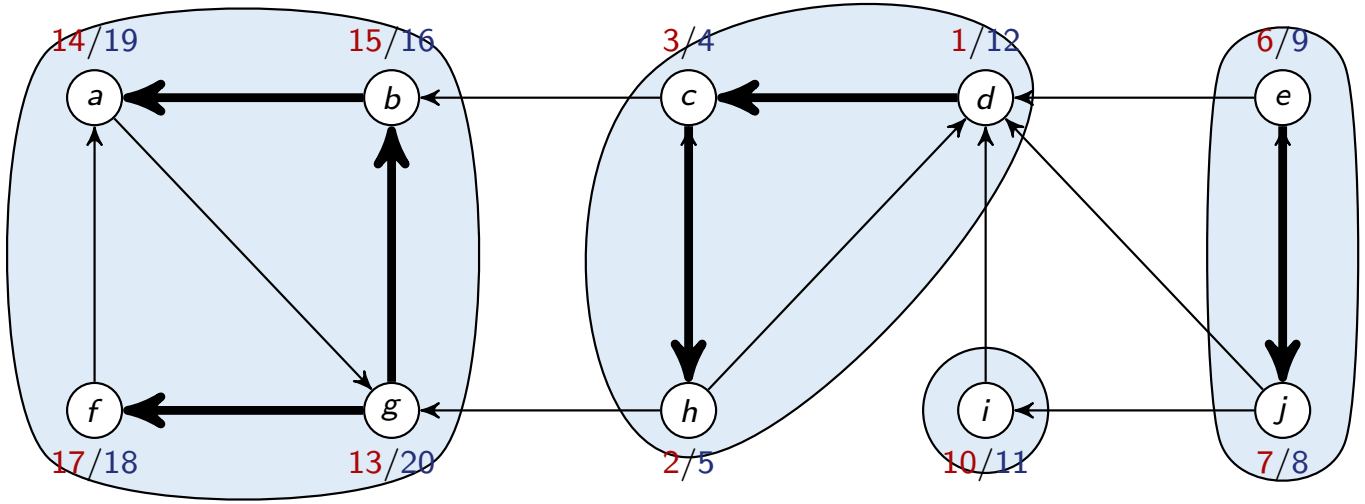

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Runtime analysis:

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Runtime analysis: Each step takes $\Theta(V + E)$ so total running time is $\Theta(V + E)$

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Why does it work?

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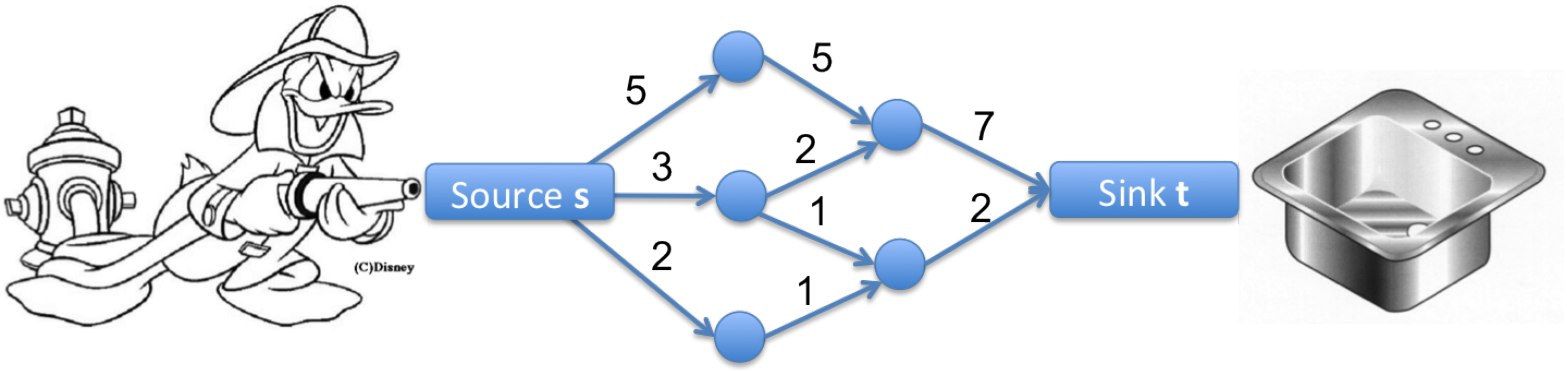
Why does it work? Intuition:

- The first DFS orders SCC's in topological order (recall G^{SCC} is acyclic)
- Second DFS then outputs the vertices in each SCC

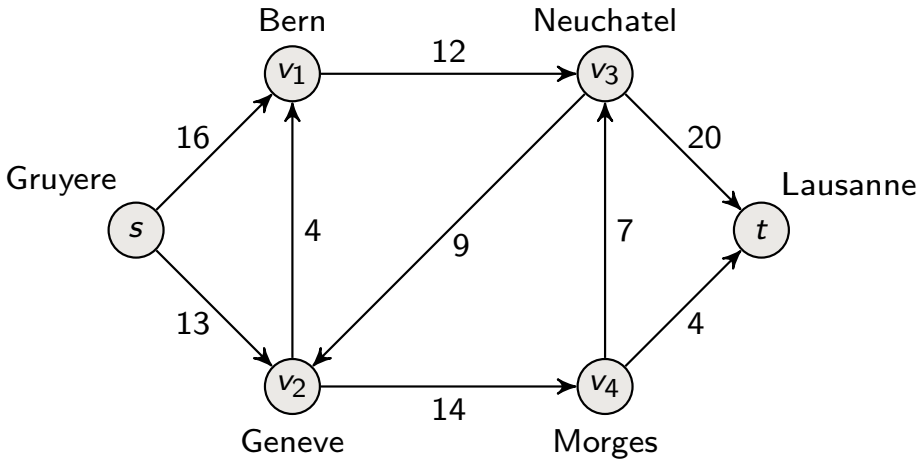
Formal proof in book



FLOW NETWORKS

Flow Network

Transfer as much cheese as possible from Gruyere to Lausanne



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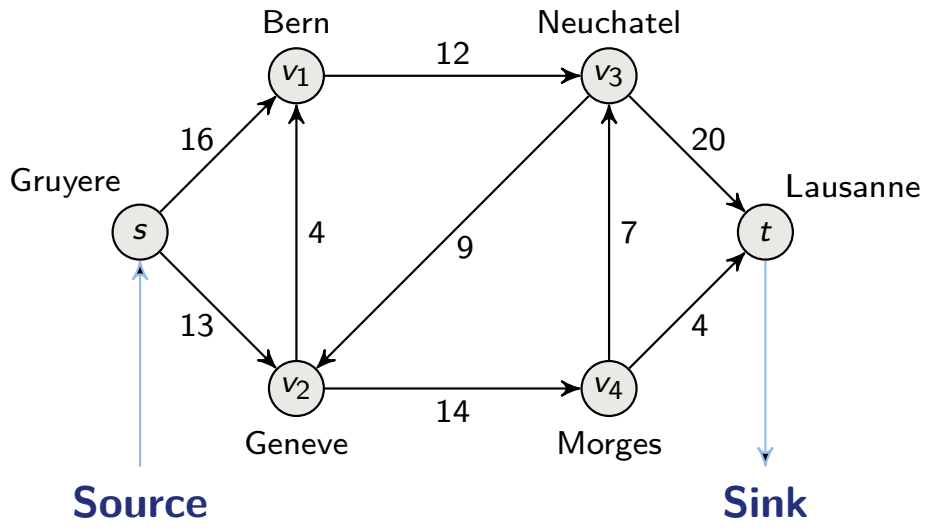

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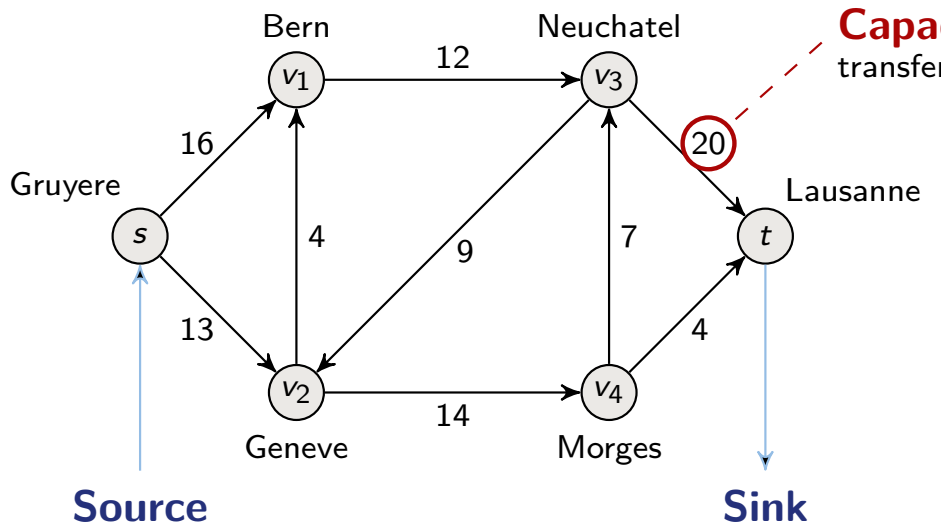

Capacity: at most 20 cheeses can be transferred from Neuchatel to Lausanne

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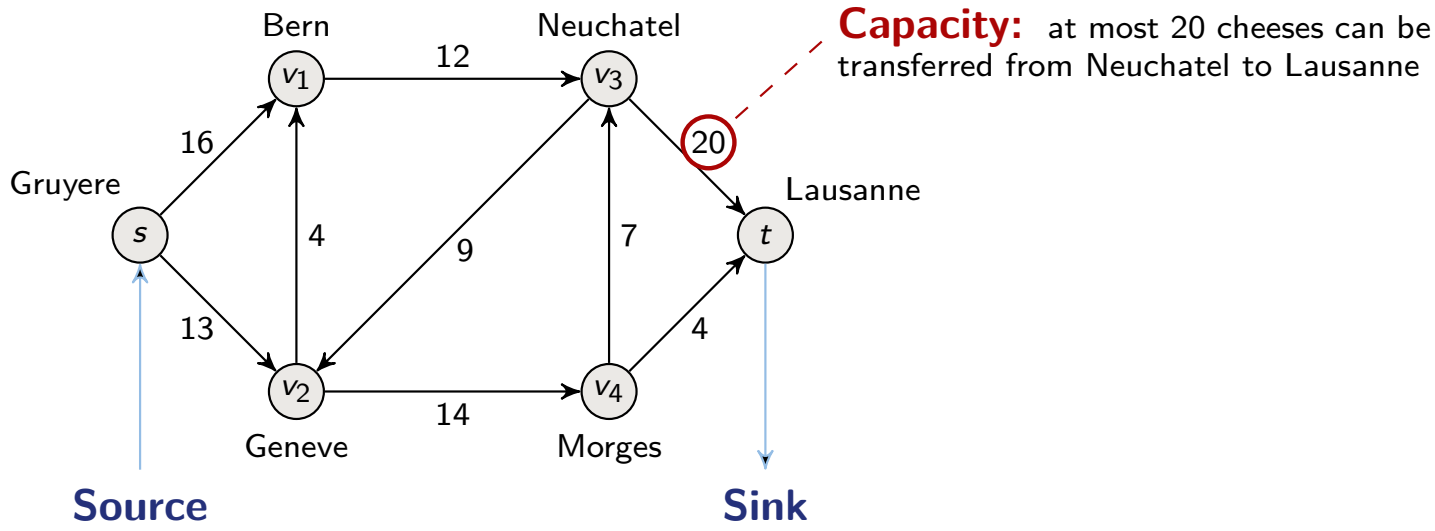

- a graph to model flow through edges (pipes)

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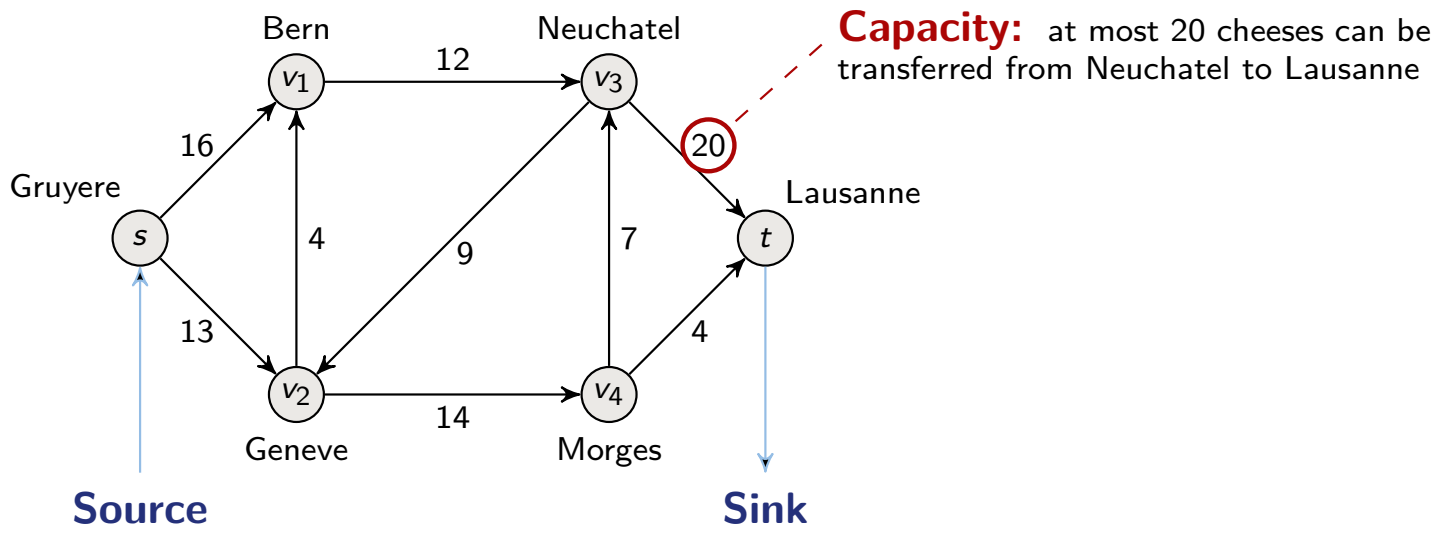

- a graph to model flow through edges (pipes)
- each edge has a capacity an upper bound on the flow rate (pipes have different sizes)

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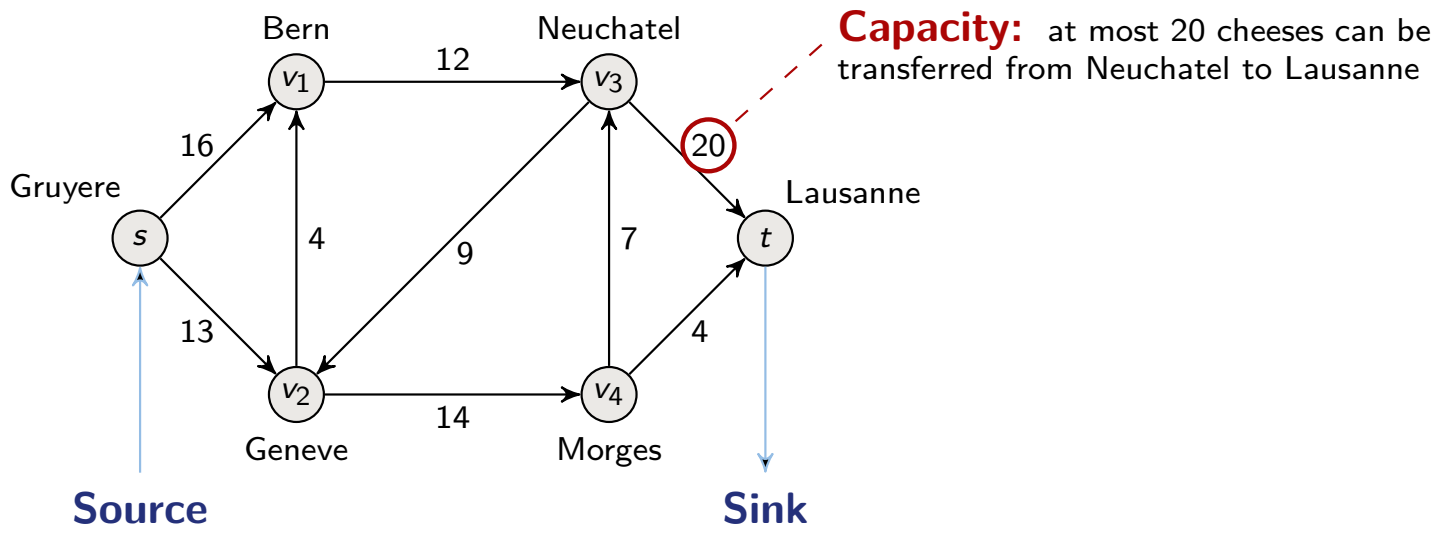
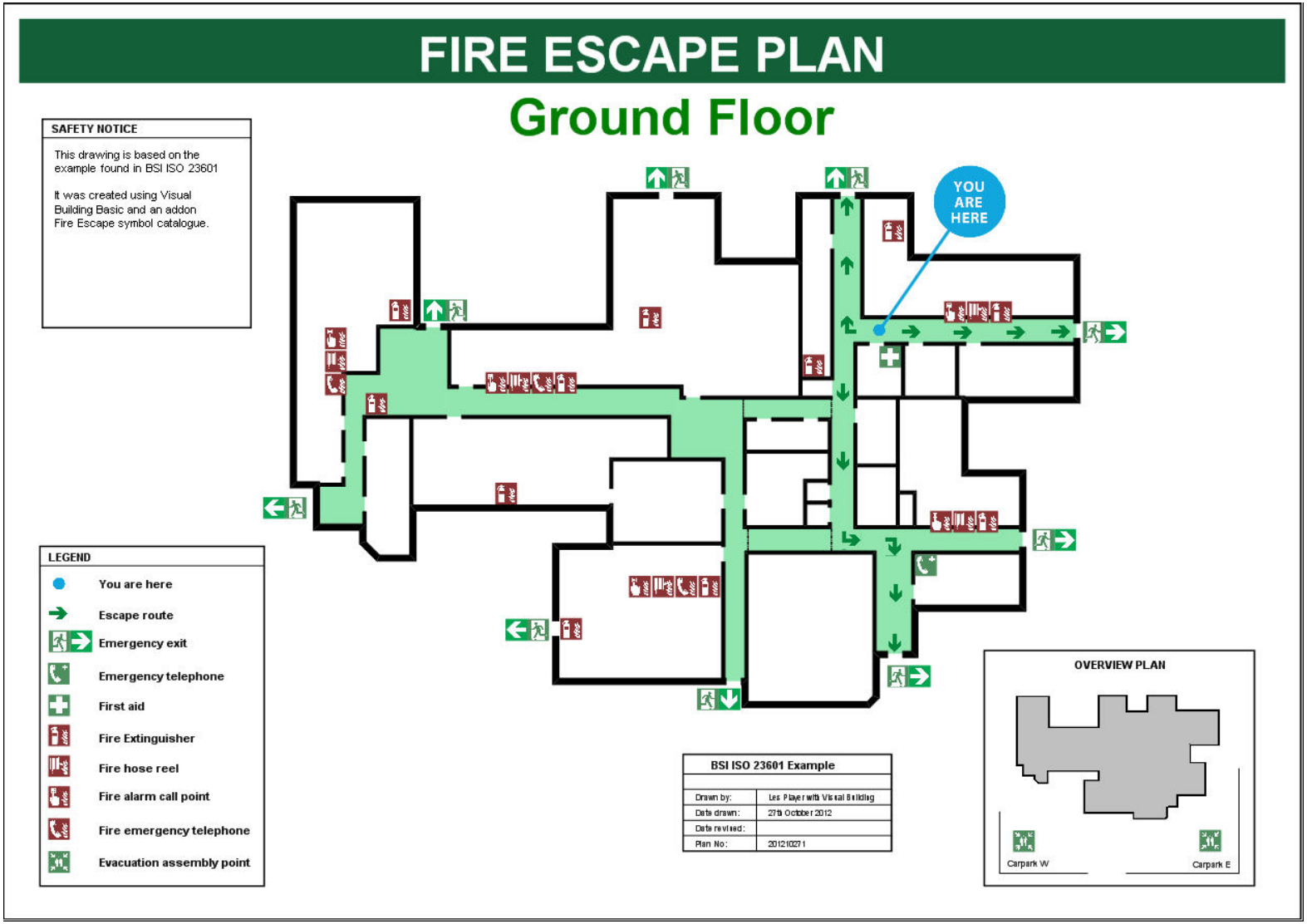


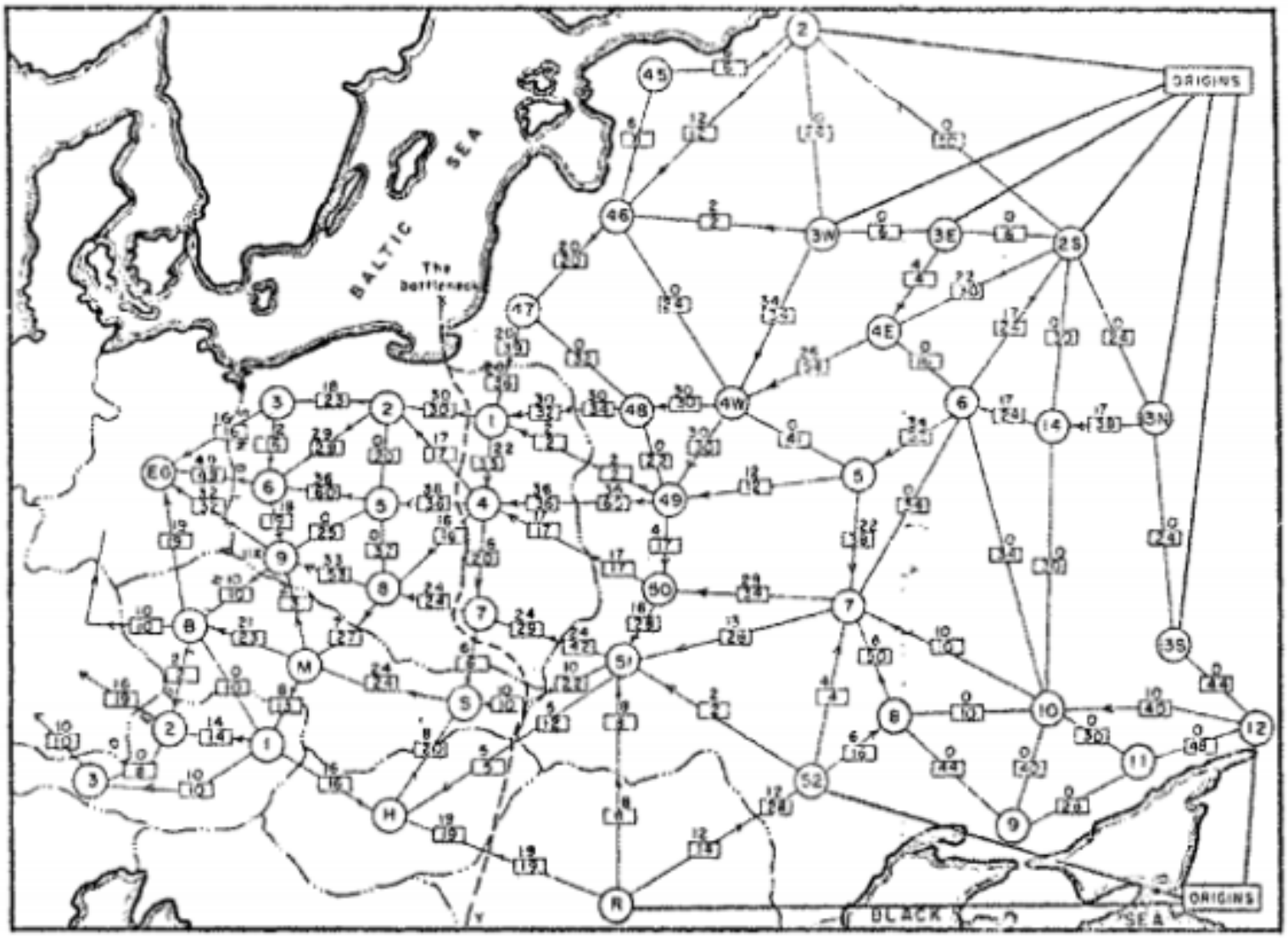
- a graph to model flow through edges (pipes)
- each edge has a capacity an upper bound on the flow rate (pipes have different sizes)
- Want to maximize rate of flow from the source to the sink

Tons of applications

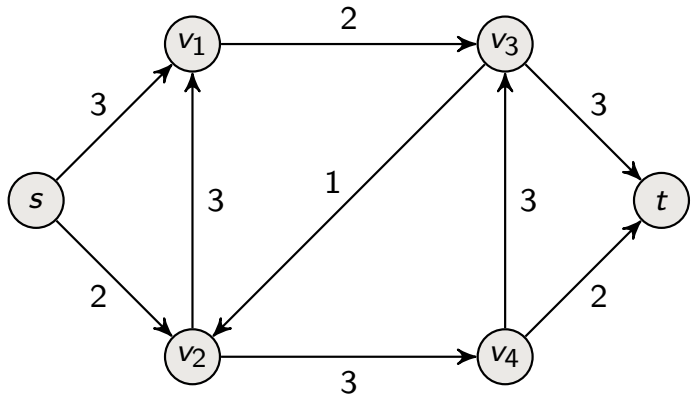
Tons of applications



Tons of applications



Flow Network (formally)



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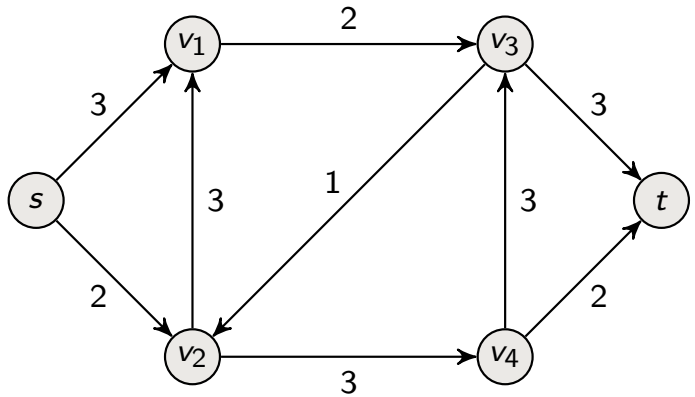
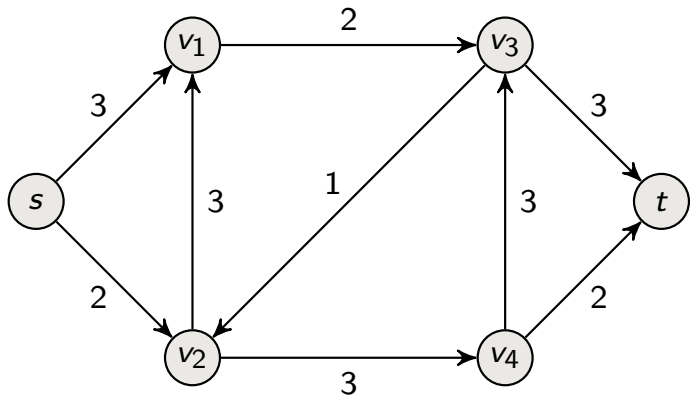


- Directed graph $G = (V, E)$

Flow Network (formally)



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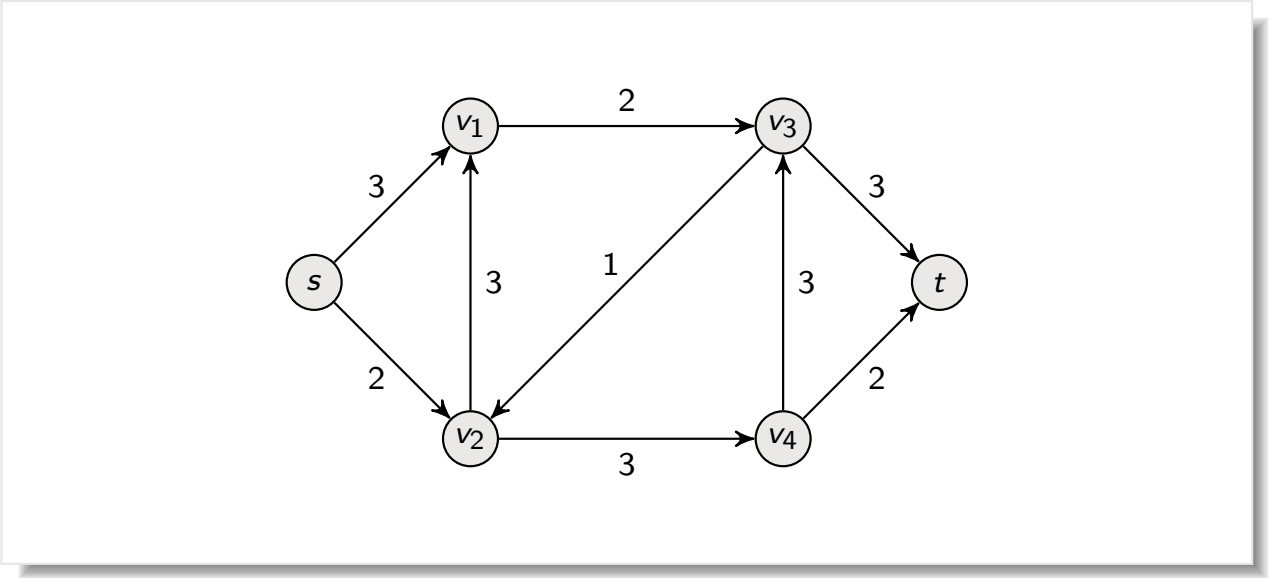

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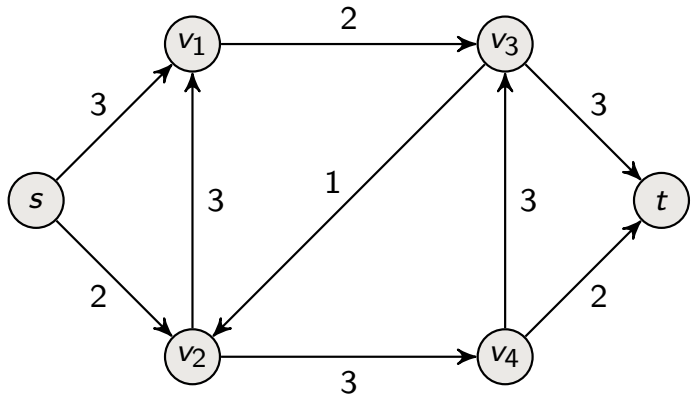

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- No antiparallel edges (assumed w.l.o.g. for simplicity)

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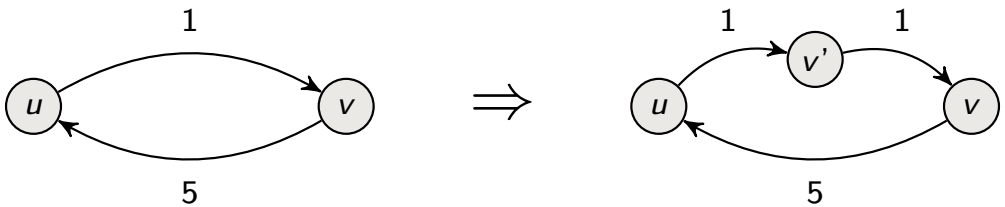

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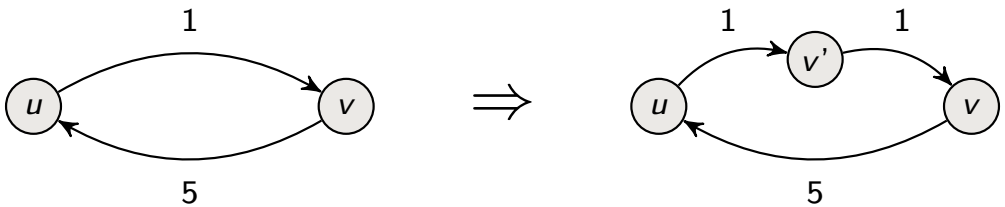

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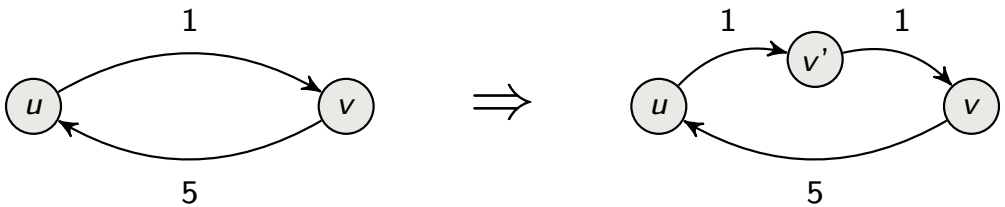

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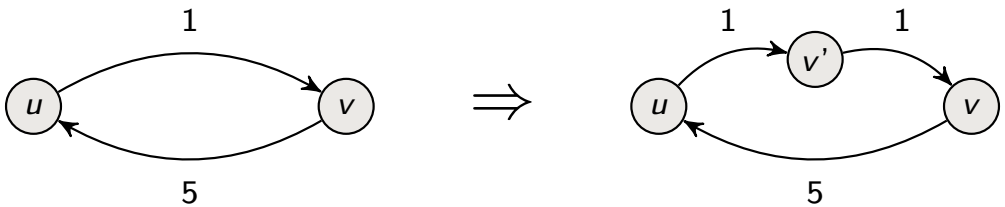

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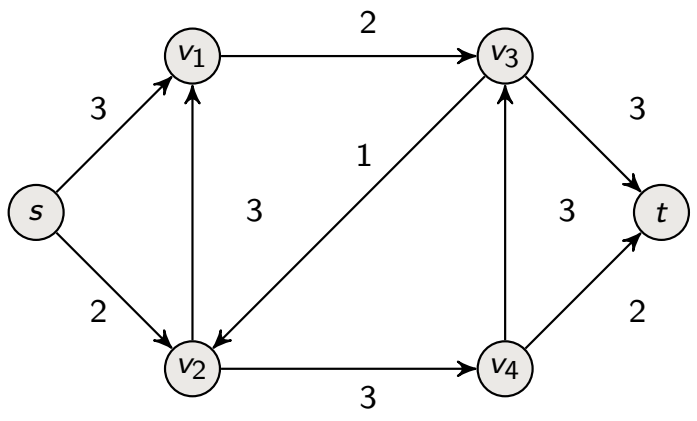
- Repeat this $O(E)$ times to get an equivalent flow network with no antiparallel edges.

Definition of a flow



A flow is a function $f : V \times V \to \mathbb{R}$ satisfying:

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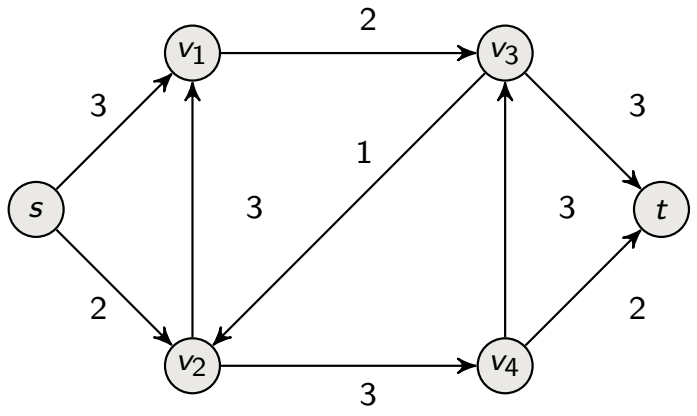

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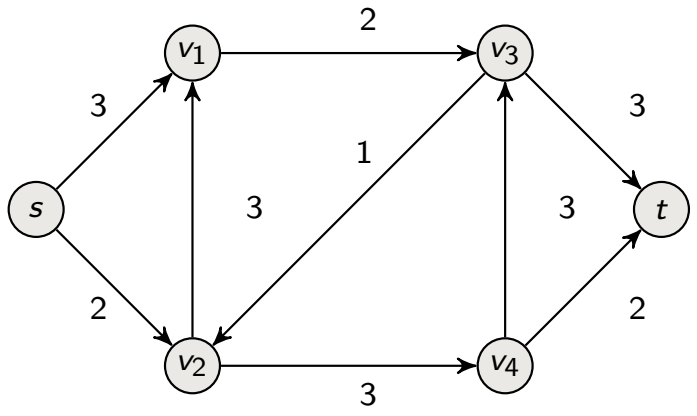

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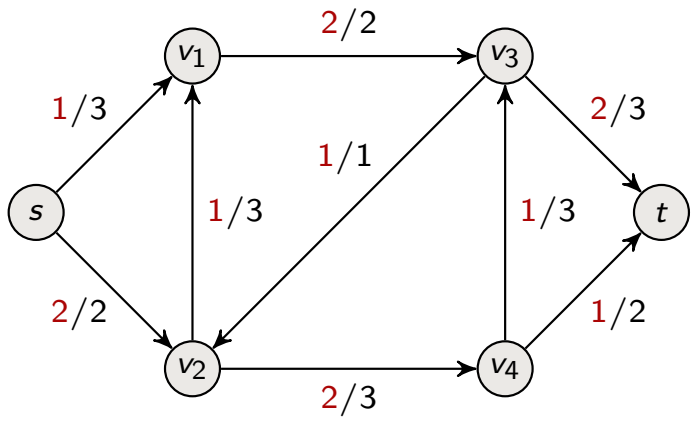

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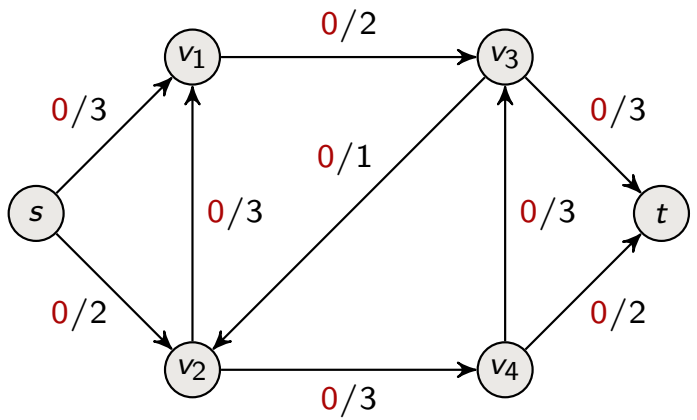

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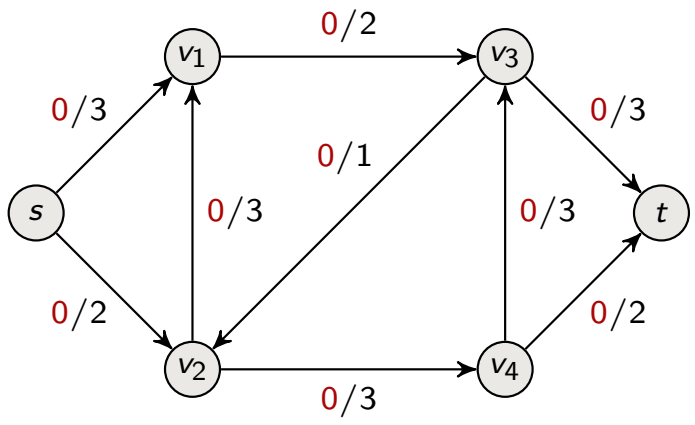

Value of a flow $f = |f|$

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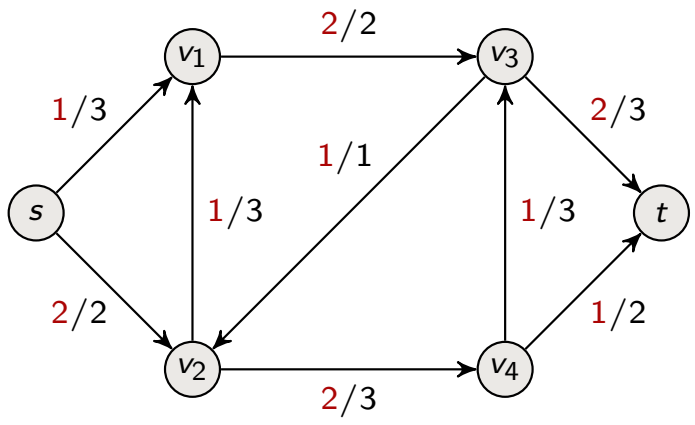

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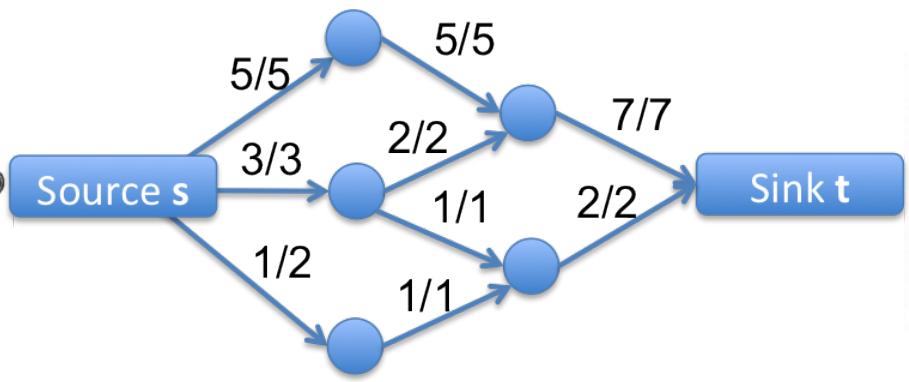

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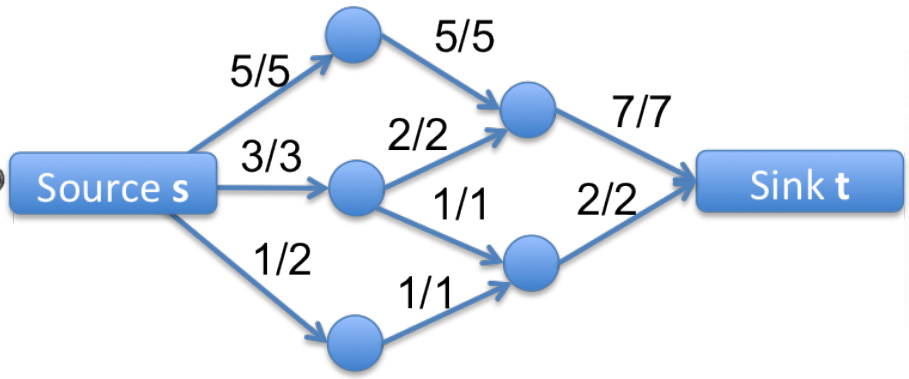
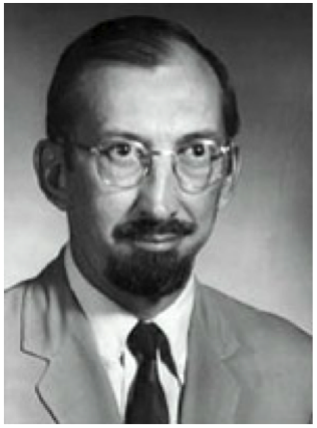


L. R. Ford, Jr. (1927-)



D, R, Fulkerson (1924-1976)

MAXIMUM-FLOW PROBLEM

Ford-Fulkerson Method

The Ford-Fulkerson Method'54

FORD-FULKERSON-METHOD(G, s, t):

1. Initialize flow f to 0
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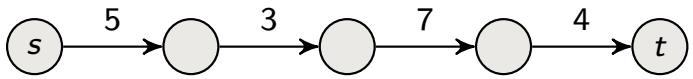

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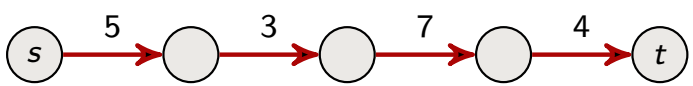

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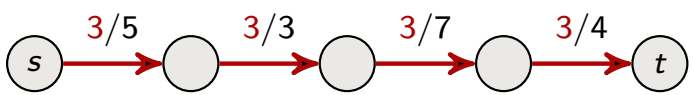

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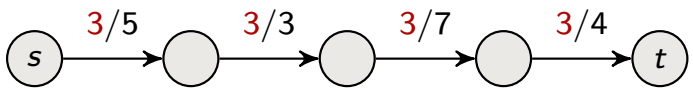

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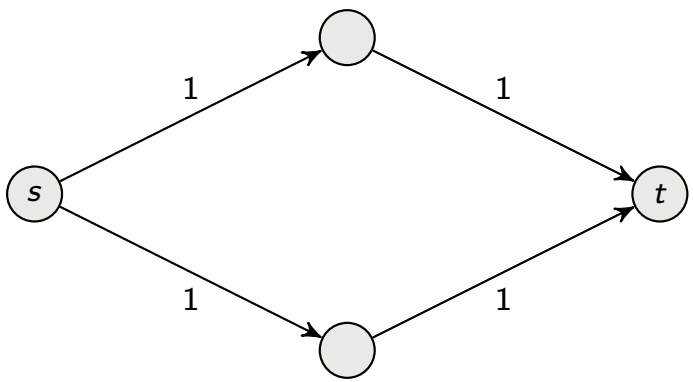

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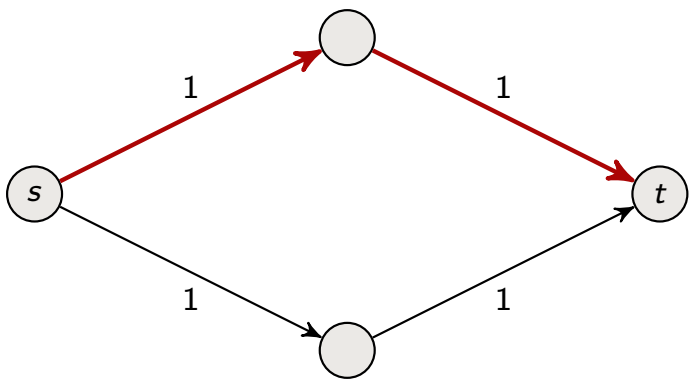

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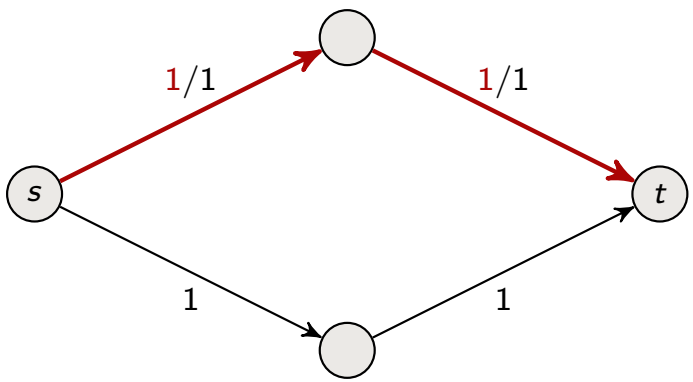

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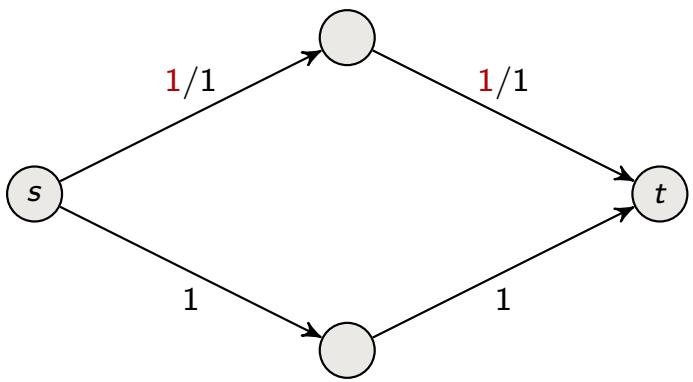

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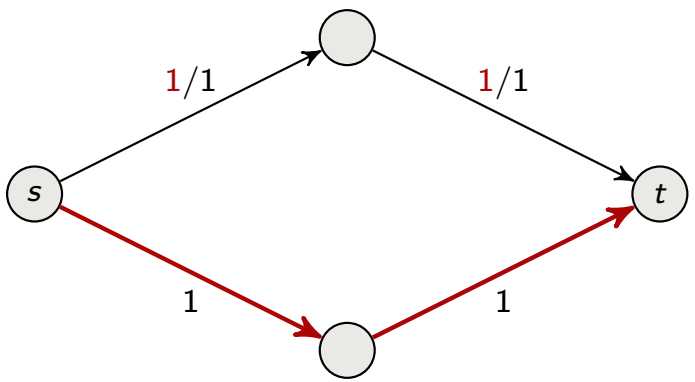

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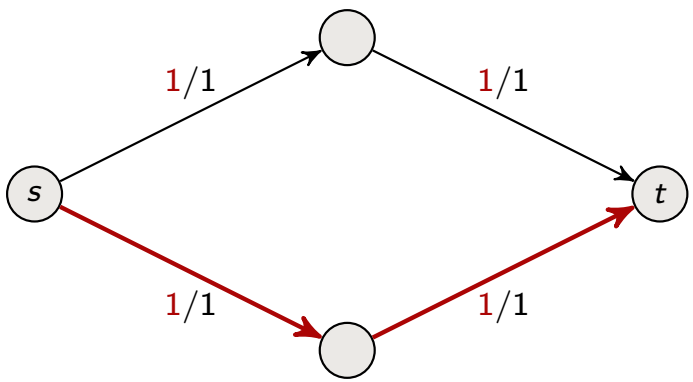

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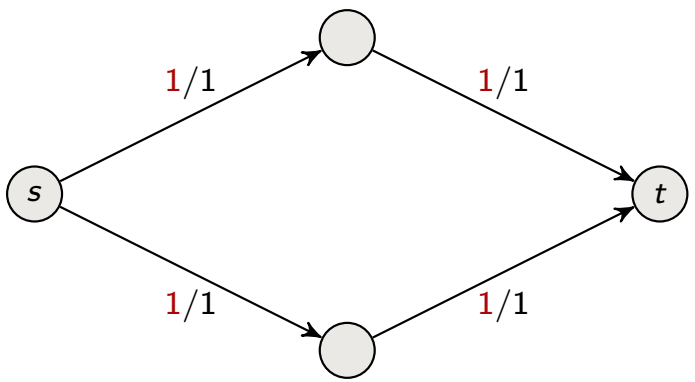

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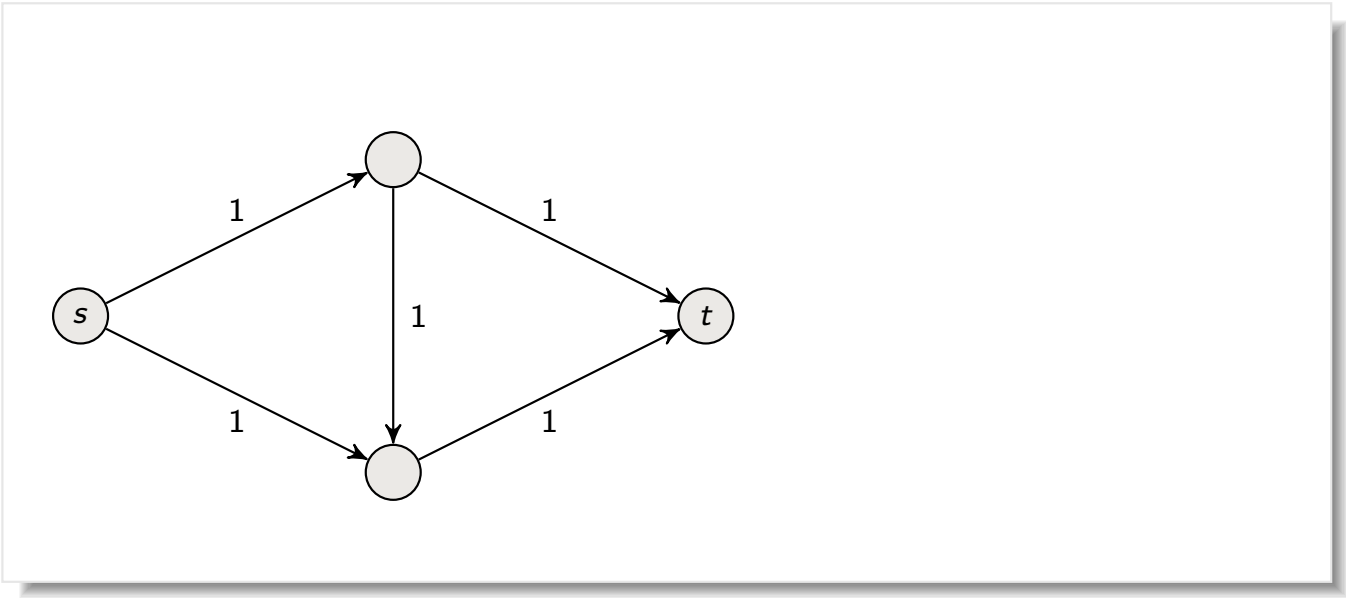

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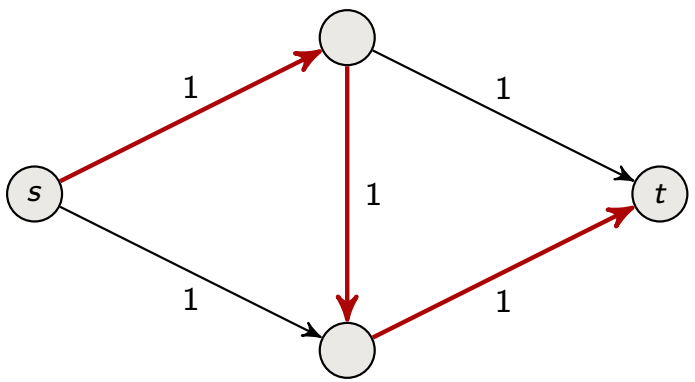

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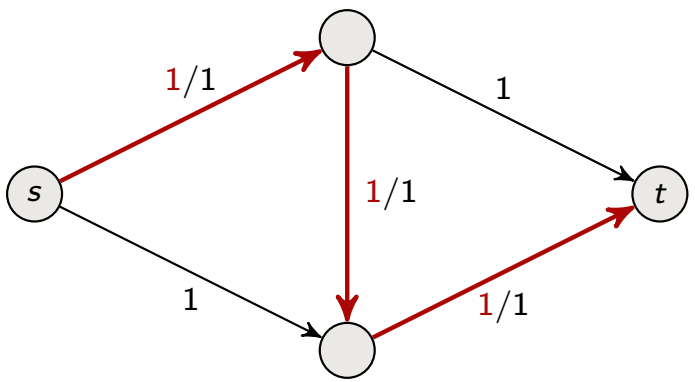

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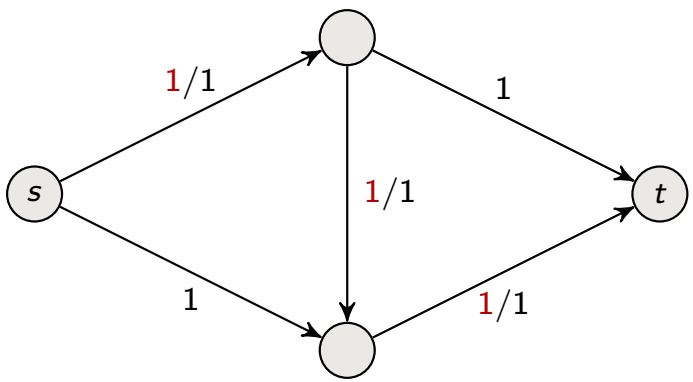

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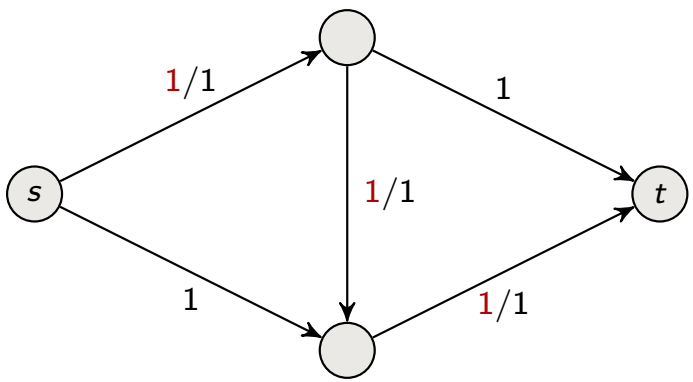

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What went wrong? How can we fix it?

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Residual network

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Examples

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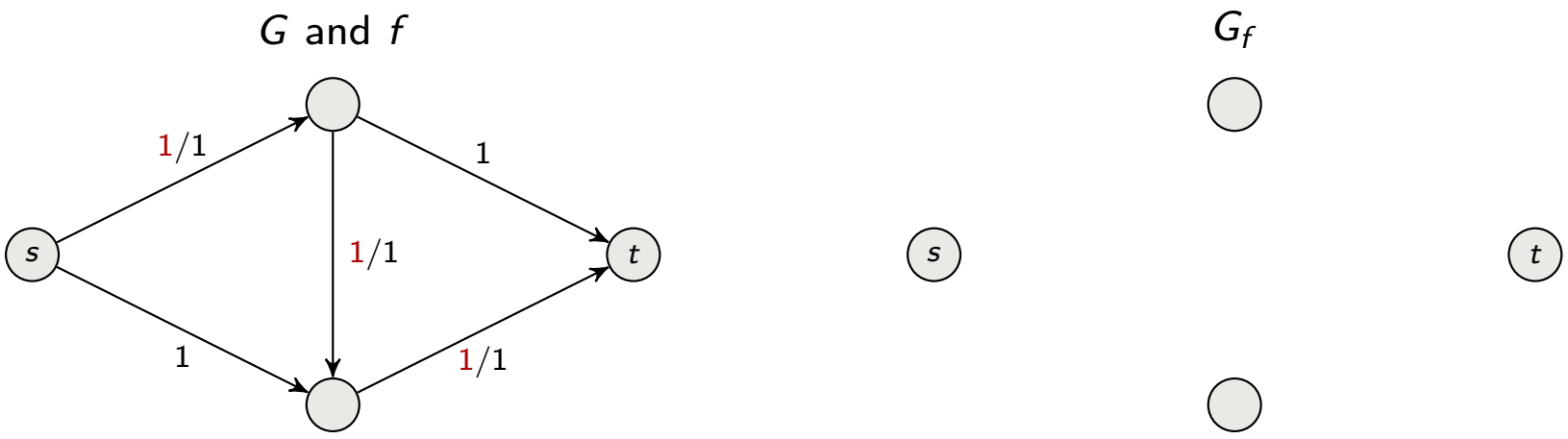

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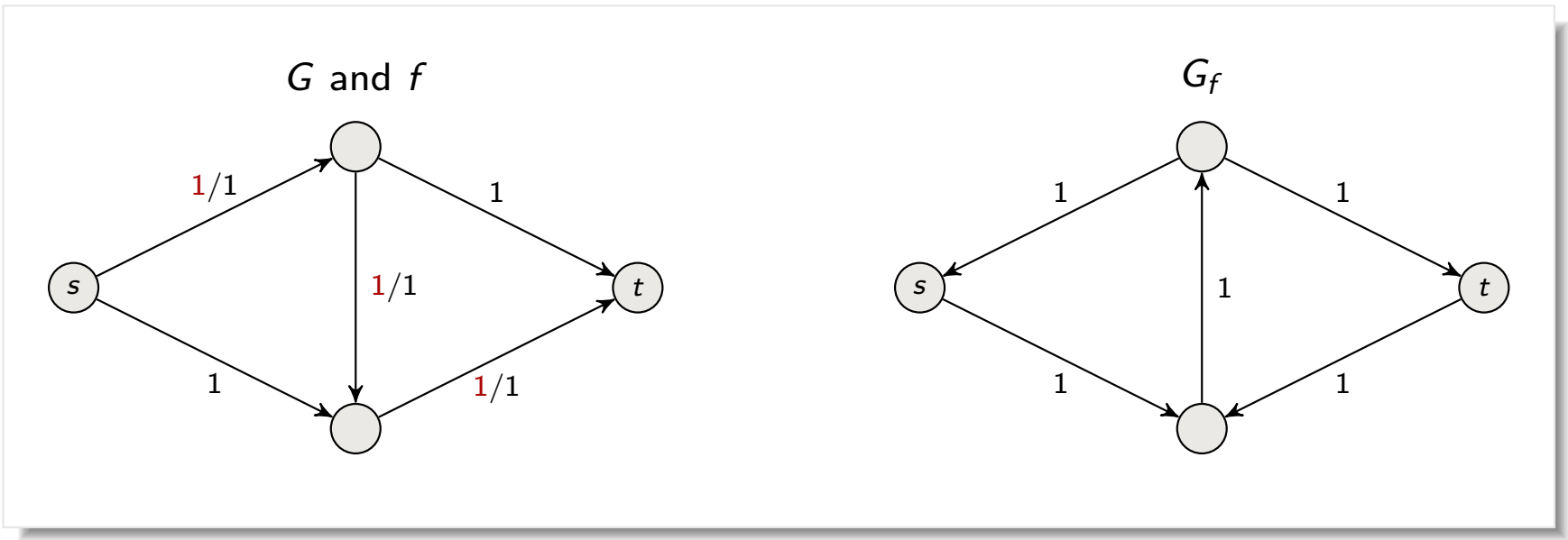
$$c_f(u, v) = \begin{cases} c(u, v) - f(u, v) & \text{if } (u, v) \in E \\ f(v, u) & \text{if } (v, u) \in E \\ 0 & \text{otherwise} \end{cases}$$



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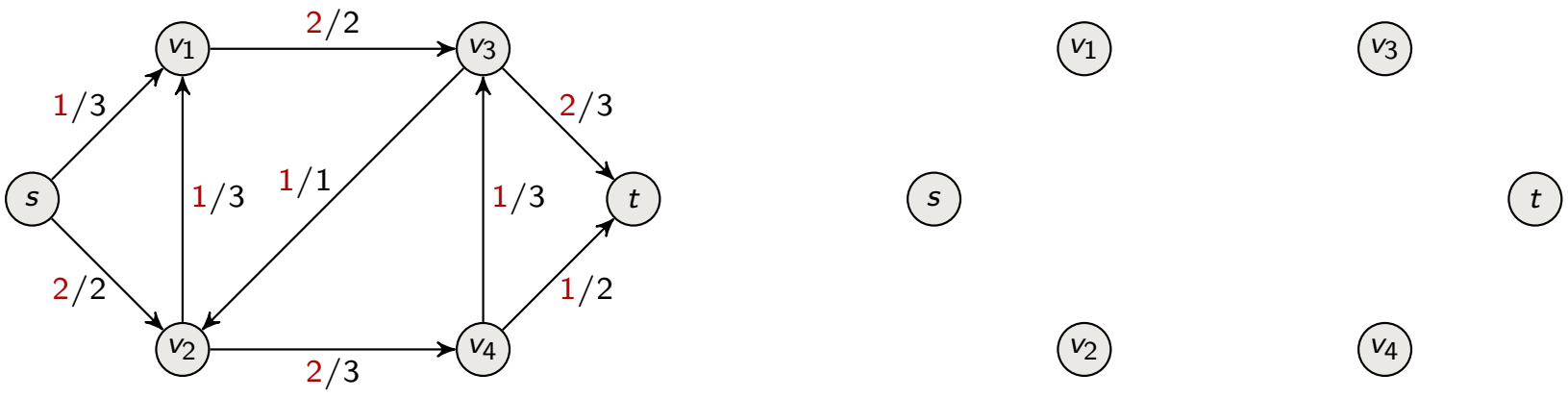

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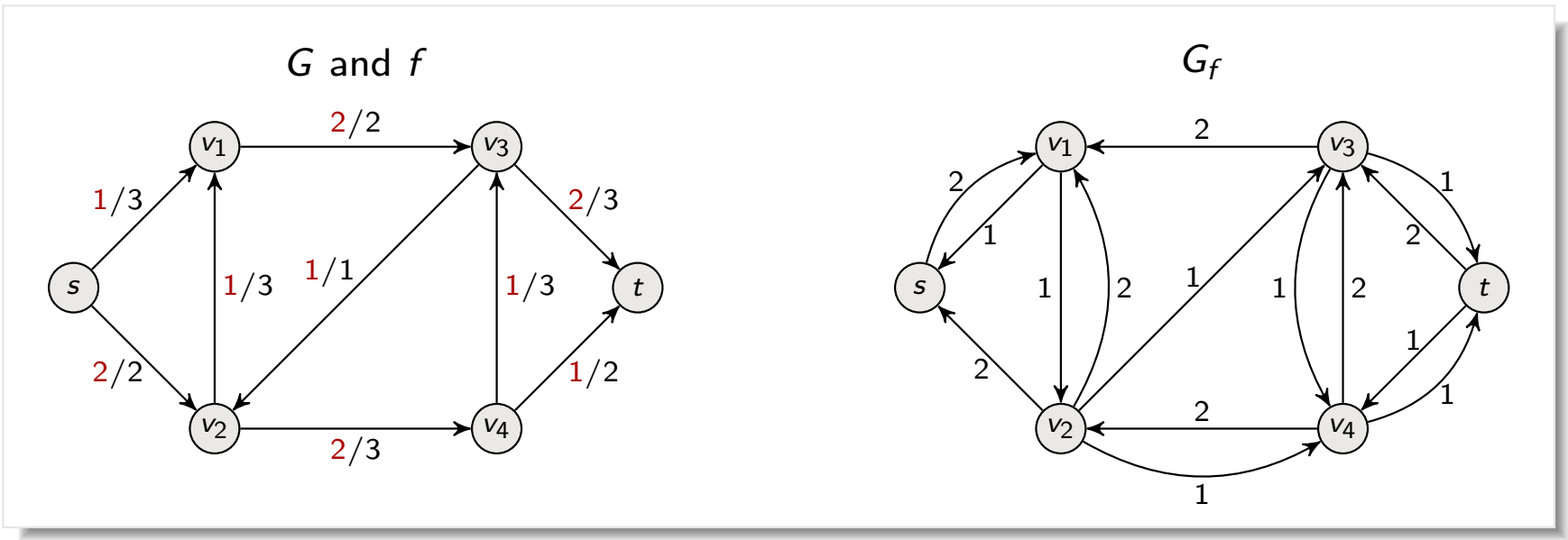

The Ford-Fulkerson Method'54

FORD-FULKERSON-METHOD(G, s, t):

1. **Initialize flow f to 0**
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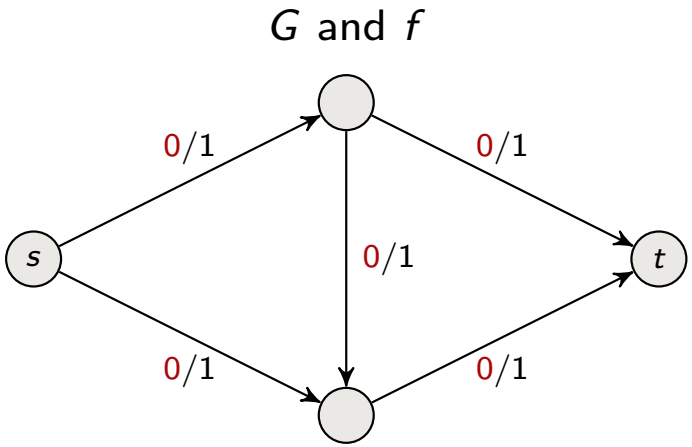

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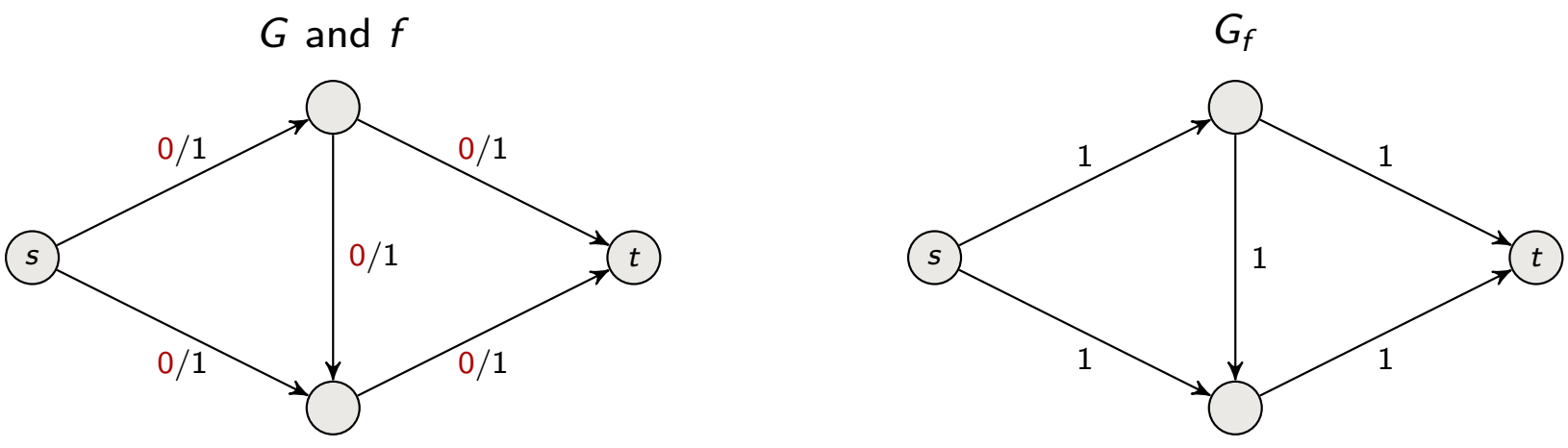
Augmenting path = simple path from s to t



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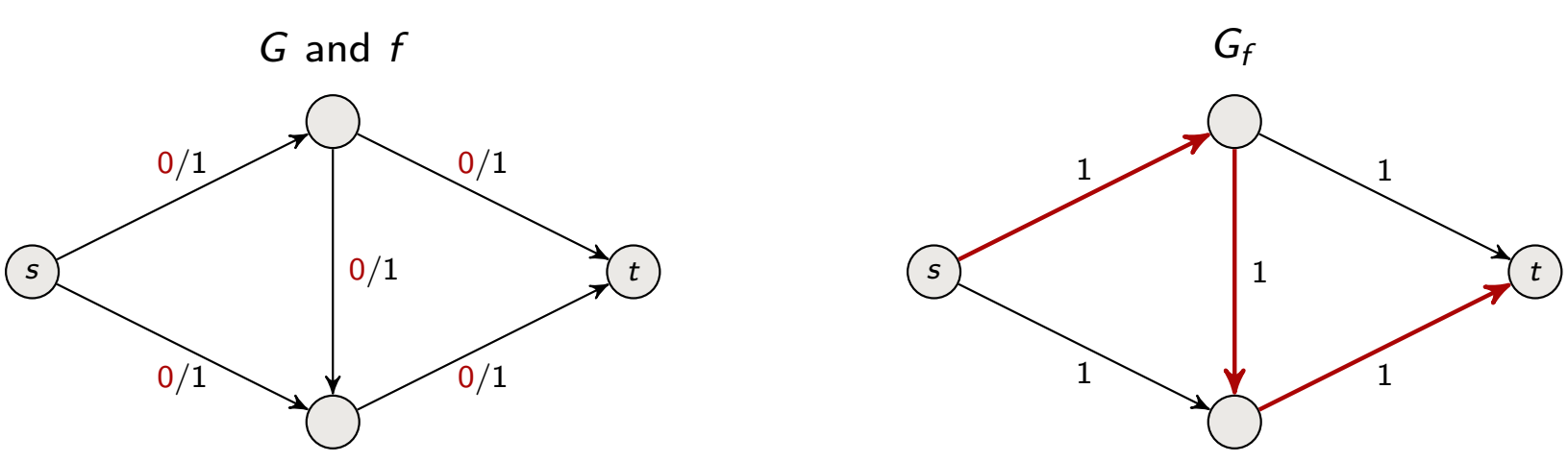
Exists augmenting path **p**



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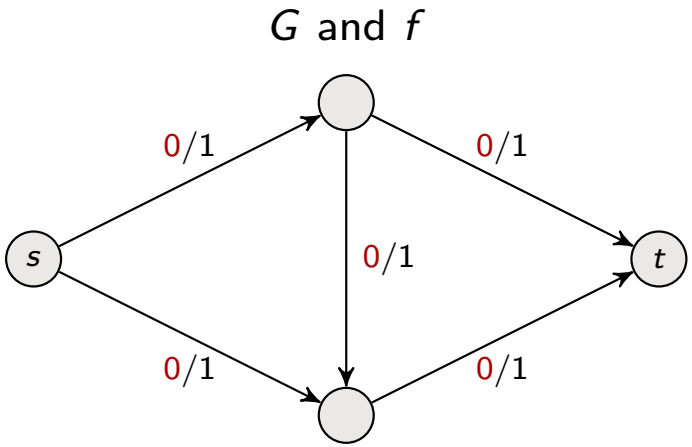
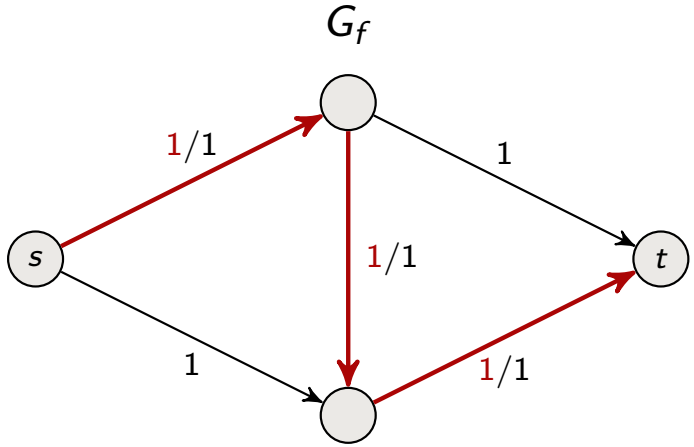
Exists augmenting path **p** with flow $\mathbf{f_p}$ of value = min capacity on **p**





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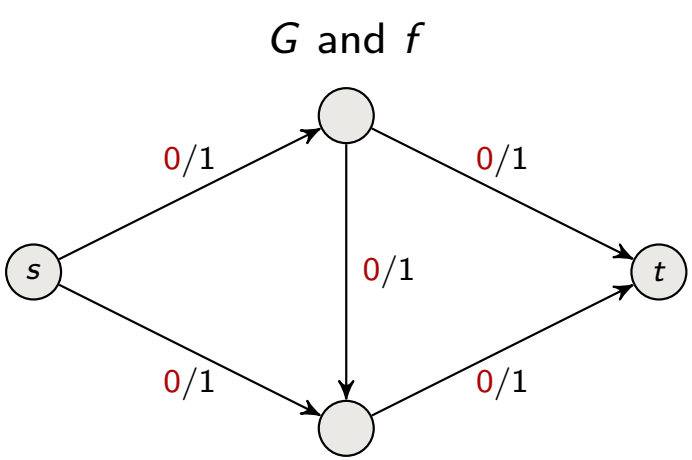

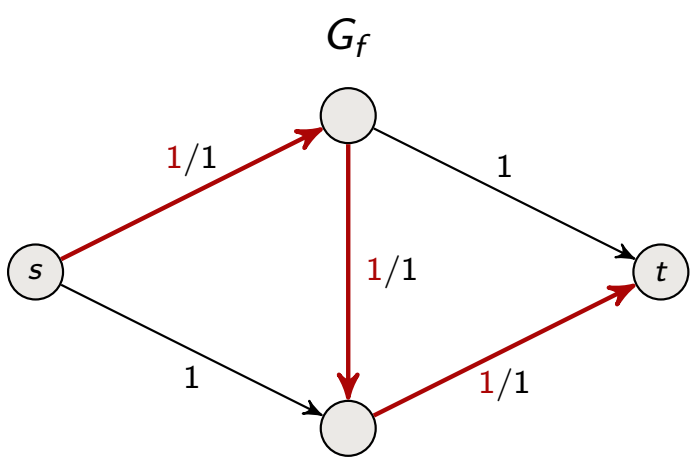

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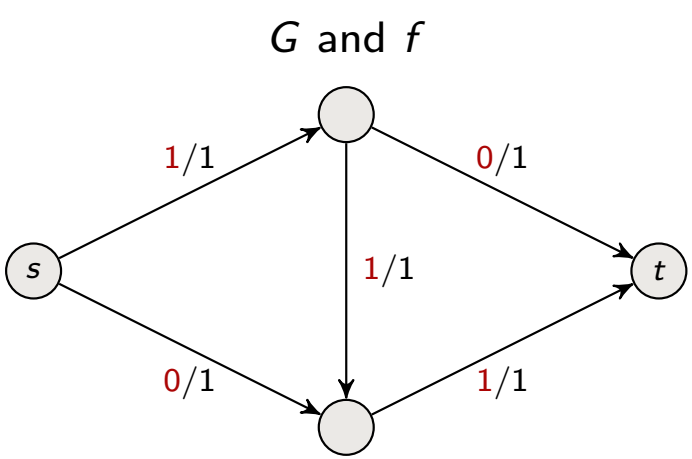

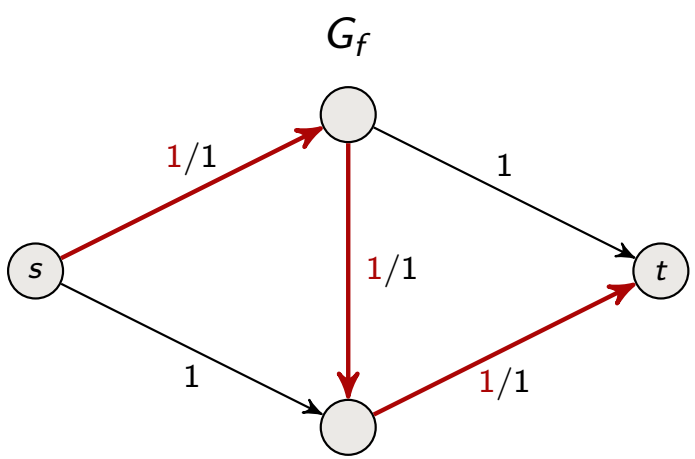

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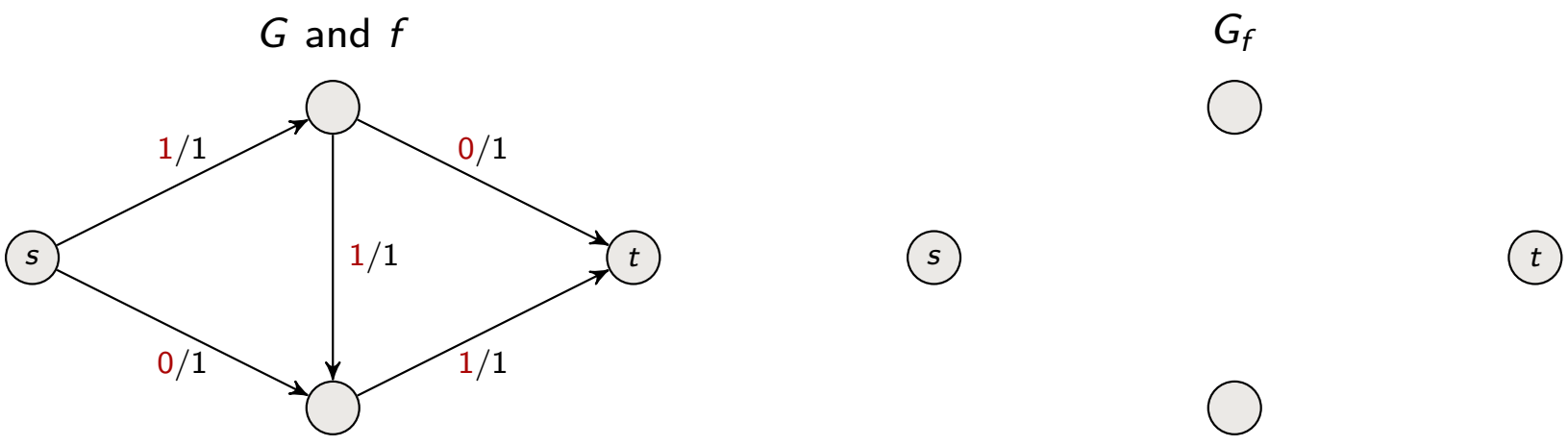

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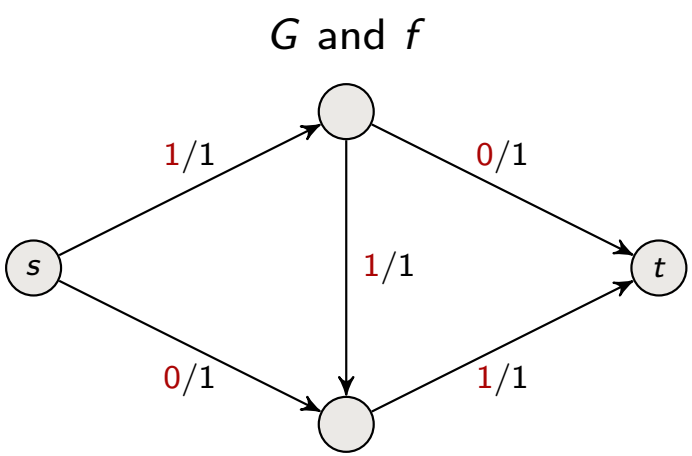

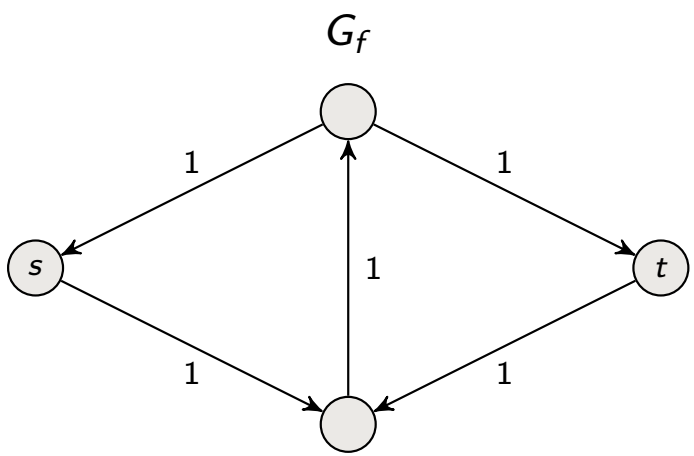

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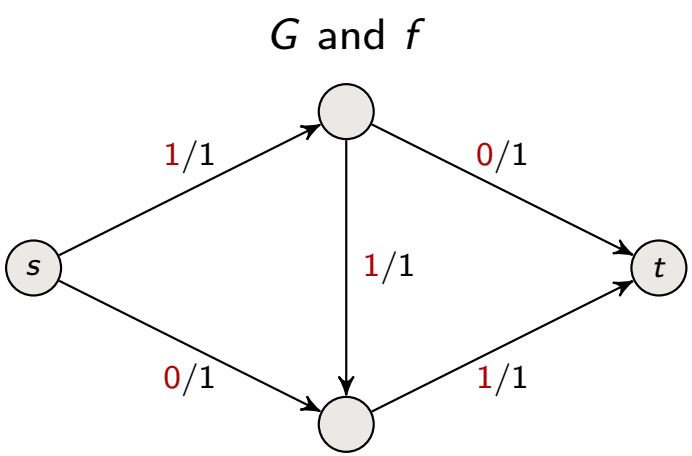

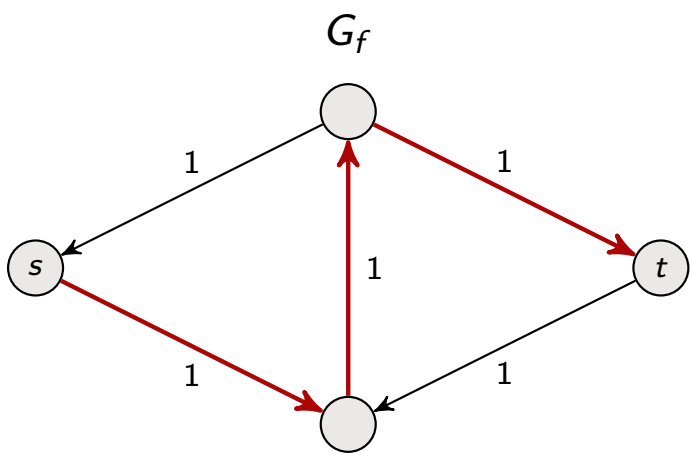

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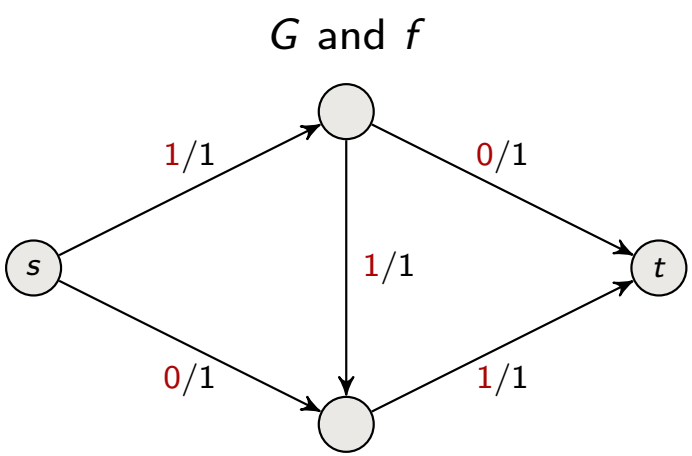

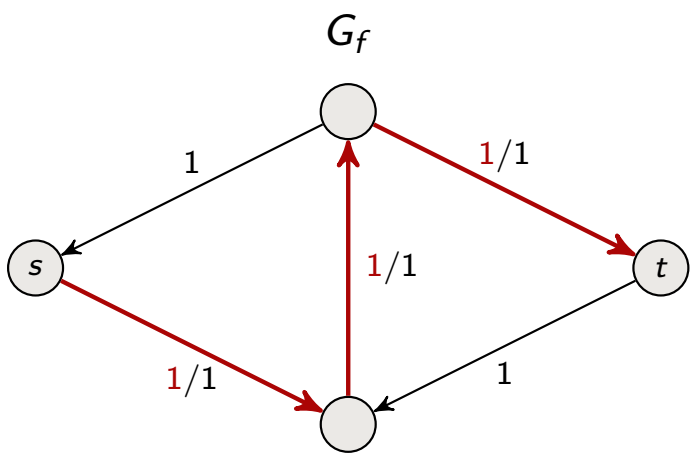

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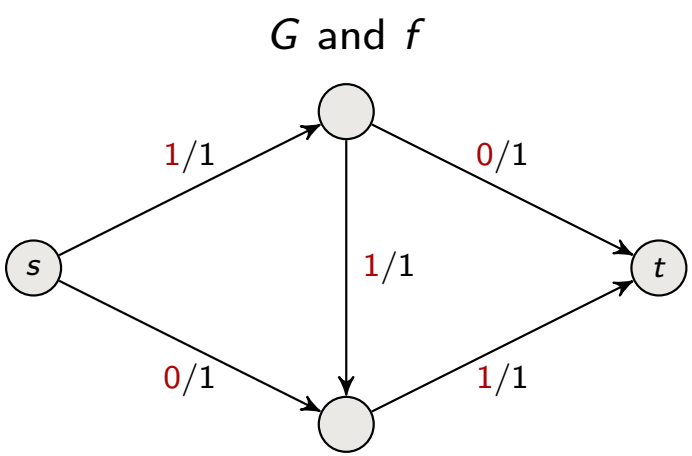

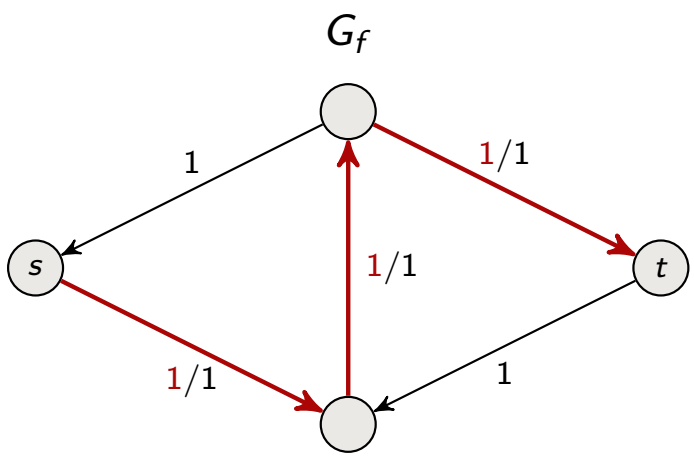

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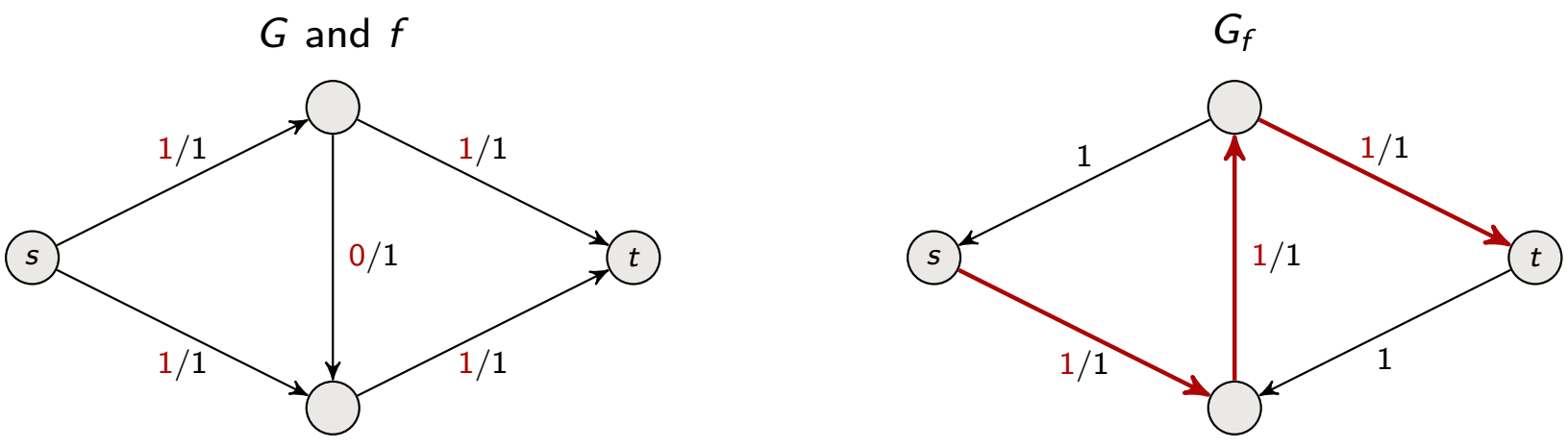

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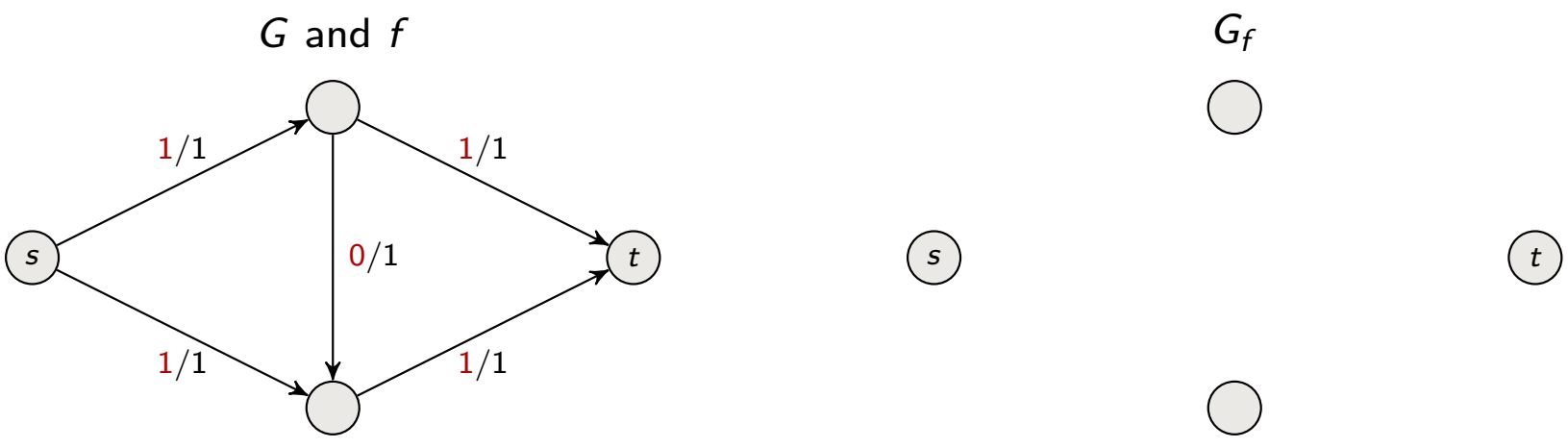

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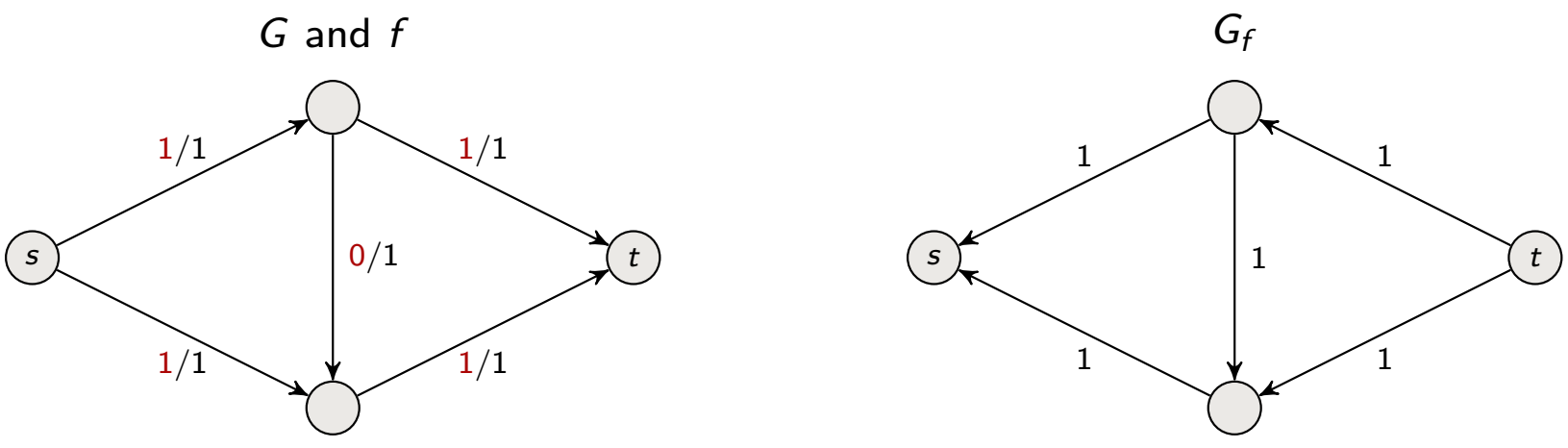

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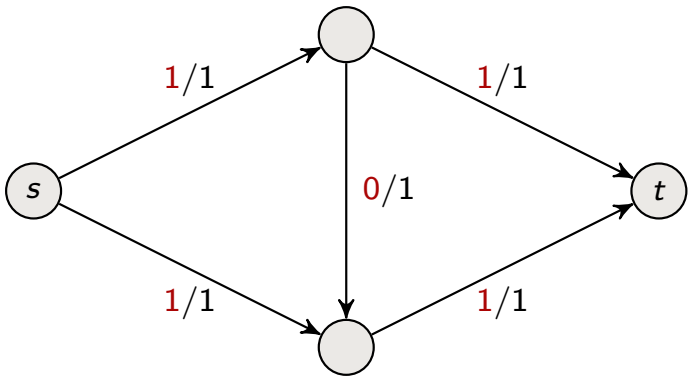
G and f



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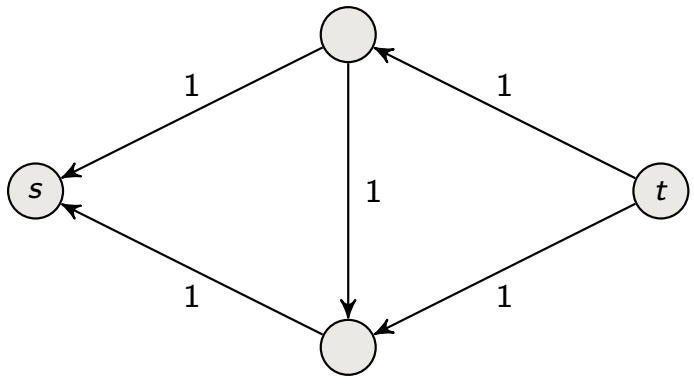

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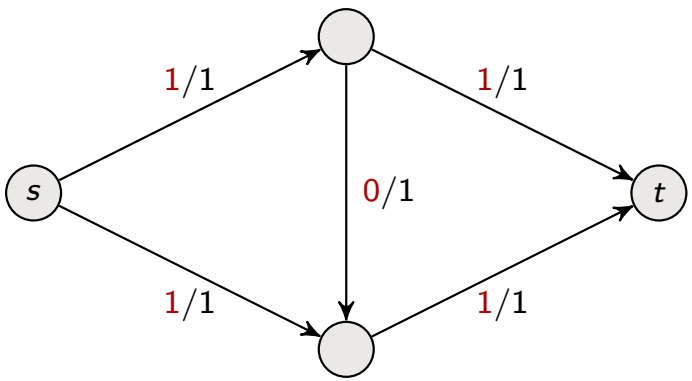

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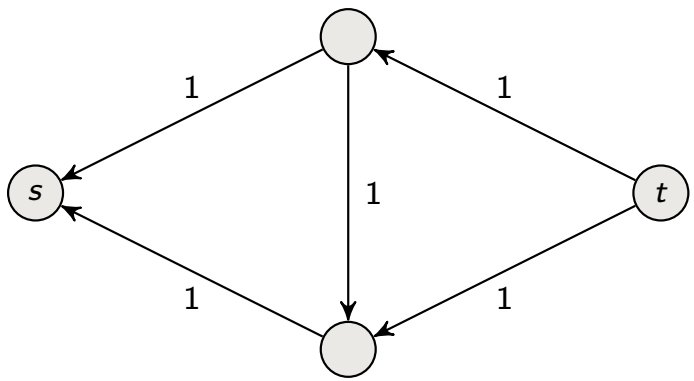
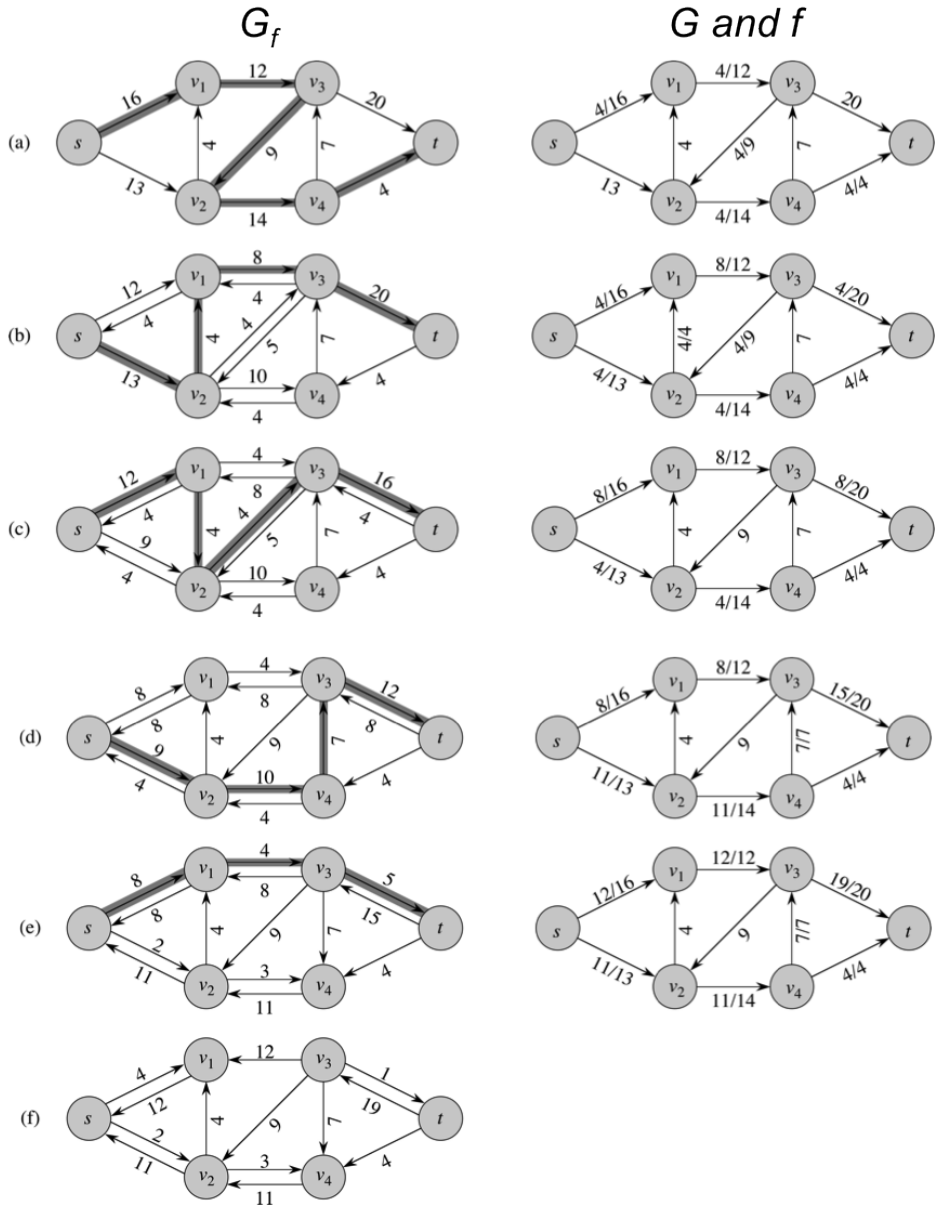


Study and understand
Example!



Summary

- Graphs fundamental object to study
- Two natural ways of traversing a graph: breadth-first search and depth-first search
- Topological sort of acyclic graphs by applying DFS and then order according to decreasing finishing times
- Strongly connected components
- Flow Networks